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# TEXT BOOK OF MECHANICS

*Designed for Colleges and Technical Schools*

By LOUIS A. MARTIN, Jr.

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# TEXT-BOOK

OF

# MECHANICS

BY

LOUIS A. MARTIN, JR.

*Professor of Mechanics, Stevens Institute of Technology*

## VOL. IV.

## APPLIED STATICS

*FIRST EDITION*

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# PREFACE

THE first two volumes of this Text-book of Mechanics (Vol. I., Statics; Vol. II., Kinematics and Kinetics) constitute a course in theoretical mechanics. These books bridge the gap between the mechanics of the course in physics and the course in applied mechanics of the technical schools.

Volume three (Mechanics of Materials), volume four now before the reader, and a projected volume dealing with applied kinetics will form a course in applied mechanics.

In this, the fourth volume, statics is applied both to structures and to machines. It thus serves as an introduction to mechanical as well as to structural engineering.

This book is intended to be a text-book — a teachable book. Moreover, to quote from the preface of the third volume, it is designed to "encourage the student to think and not to memorize, to do and not simply to accept something already done for him"; it nevertheless "furnishes sufficient material in the way of explanation and example so that he will not become discouraged."

I take this opportunity to acknowledge my indebtedness and to express my thanks to my wife, Alwynne B. Martin, for assistance in the preparation of the manuscript and in the reading of the proof, and to Prof. R. F. Deimel and Mr. G. G. Freygang for reading both manuscript and proof.

L. A. M., JR.

CASTLE POINT, HOBOKEN, N. J.
*November*, 1912.

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# APPLIED STATICS

## CHAPTER I

## WORK AND ENERGY

### SECTION I

### INTRODUCTION

THE work done by a constant force is defined as the product of the force and the projection of the displacement of its point of application upon the line of action of the force.

If the projected displacement agrees in direction with the direction of the force the work is considered positive, and thus if the projected displacement takes place in the opposite direction to that in which the force acts the work done must be considered negative.

The **units** employed by English-speaking engineers are as follows:

The **unit of force** is the attraction exerted by the earth upon a certain piece of platinum when at a latitude of 45° and at the sea level. This piece of platinum is marked "P. S., 1844, 1 lb.," and is carefully preserved in London. This unit of force is called the pound.

The **unit of displacement** is the foot, being one-third of the distance between two marks upon a platinum bar (known as the standard yard), also preserved in London.

The **unit of work** is thus the foot-pound, that is, the work done whenever a force of one pound is displaced along its line of action thru a distance of one foot.

Whenever a free body is displaced by a couple, angular displacement occurs about the mass-center of the body and the **work done by the couple** is then readily measured by the product of the moment of the couple (or its torque) and the angular displacement produced.

The **unit of angular displacement** is the radian; an angular displacement of 360° or one revolution is equal to $2\pi$ radians.

That this method of calculating the work done by a couple follows from the definition of work may be demonstrated as follows:

In Fig. 1 consider a free body acted upon by a couple whose torque is $Fa$. Let $C$ be the mass-center of the body. The body will then be displaced angularly (rotated) in the direction of the curved arrow. If the body suffers a displacement of one revolution the force $F$, acting in the direction of the displacement, will at each instant be displaced along an element of a circle whose radius is $(a + x)$ and the work done will thus be $F\ [2\pi\,(a + x)]$. In the meantime the other force $F$ performs work equal to $-F\ (2\pi x)$. The total work done by the couple is thus $2\pi F\ (a + x) - 2\pi Fx = 2\pi Fa$. But $Fa$ is evidently the torque of the couple to be measured in pound-feet, while $2\pi$ is the angular displacement of the body measured in radians, and $2\pi Fa$, the work done, is measured in foot-pounds.

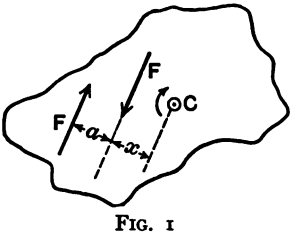

FIG. 1

In Fig. 2 consider the work done by gravity upon the wheel and axle during one revolution, neglecting friction and assuming rotation in the direction indicated by the curved arrow.

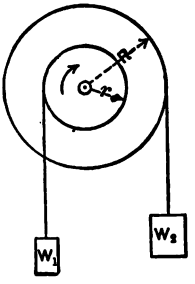

FIG. 2

Here the attraction of gravity produces tensions of $T_2$ and $T_1$ pounds in the ropes attached to $W_2$ and $W_1$ respectively. The moment or torque of $T_2$ is $T_2R$ and that of $T_1$ is $T_1r$, while the angular displacement of the wheel and axle upon which these torques act is $2\pi$ radians.

The work done is therefore

$$2\pi T_2R - 2\pi T_1r.$$

EXERCISE 1. (a) Show the wheel and axle of Fig. 2 as a "free body" and indicate clearly the couples involved in the above discussion.

(b) Calculate the work done by gravity upon the weights $W_2$ and $W_1$ and interpret the negative sign. Is this work equal to the work $2\pi T_2R - 2\pi T_1r$ calculated above?

EXERCISE 2. A fly-wheel, external radius $r$ feet, revolves at $n$ r.p.m. It is brought to rest by a frictional force of $F$ pounds exerted on its rim by means of a block of wood. How much work is done by this frictional force during the first 3 revolutions and also during the last 3 revolutions of the fly-wheel? How much work does the fly-wheel do during any 5 revolutions after the frictional force is applied? Draw the fly-wheel as a free body and show the couple involved in the above calculation.

**The work done by a fluid** enclosed in a cylinder and moving a piston closing one end of the cylinder (as in a gas engine) is conveniently measured by the product of

the change in volume (or the volume swept thru by the piston) and the pressure exerted by the fluid. The change in volume is measured in cubic feet, the pressure exerted by the gas in pounds per square foot, and the work done is then expressed in foot-pounds.

Thus, if before expansion the volume of the enclosed gas is $v_1$ cubic feet and after expansion $v_2$ cubic feet and the pressure exerted is constant and equal to $p$ pounds per square foot, then the work done is

$$(v_2 - v_1)\ p \text{ foot-pounds.}$$

EXERCISE 3. (a) Prove this formula assuming the area of the piston to be $A$ square feet.

(b) How would the result be interpreted when $v_2 < v_1$?

**Power** is defined as the rate at which work is done. A unit of power would be the foot-pound per second. English-speaking engineers employ the **horse-power**, defined as 550 foot-pounds per second or 33,000 foot-pounds per minute, as the practical unit of power.

## SECTION II

## WORK DONE BY CONSTANT FORCES

### Dynamometers

Dynamometers are devices for measuring the power delivered by prime movers (steam or gas engines, turbines, etc.) or consumed in driving machinery.

In reality dynamometers are measurers of force and not of energy, and a further reading of speed, involving displacement and time, must be taken to calculate the power delivered or absorbed.

Dynamometers may be divided into two general classes:

(a) **Absorption** dynamometers, which waste the energy they measure.

(b) **Transmission** dynamometers, which measure the energy while transmitting it with only slight unavoidable losses due to friction.

### Absorption Dynamometers

**The Prony Brake.** — One form of Prony brake, Fig. 3, consists of an iron hoop, to which are fastened wooden blocks. These blocks rub over the rim of an iron wheel; the normal pressure and therefore the friction between the blocks and the wheel can be adjusted by means of the bolt at $A$. The wheel is driven by the prime mover whose output of energy is to be measured. The brake, being fastened to a fixed support by means of the hook and the spring balance $T$ to prevent its rotating with the wheel, absorbs this energy and dissipates it as heat.

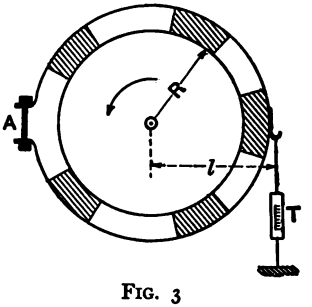

FIG. 3

To measure the energy absorbed the reading of the spring balance and the simultaneous count of the revolutions per minute of the wheel must be taken.

The wheel evidently exerts a torque upon the brake, and this torque must equal the product of the reading of the spring balance, $T$ pounds, and the arm, $l$ feet (measured from the center of the shaft of the wheel normal to the line of action of $T$), for the brake is in equilibrium.

The brake therefore exerts a torque equal to $Tl$ upon the wheel in a direction opposed to its motion and absorbs $Tl\,(2\,\pi)$ foot-pounds of work per revolution of the shaft. If the shaft makes $N$ r.p.m. then the horse-power absorbed by the brake or delivered by the prime mover is $\frac{2\,\pi NTl}{33{,}000}$.

EXERCISE 4. Show the brake illustrated in Fig. 3 as a free body and explain how the wheel exerts its torque on the brake.

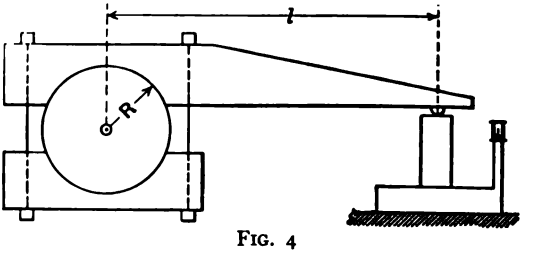


FIG. 4

EXERCISE 5. (a) Find the H.P. absorbed by the brake shown in Fig. 4 if the "effective pressure" upon the platform-scales is $P$ pounds and the shaft makes $N$ r.p.m.

(b) How would you find this "effective pressure" experimentally?

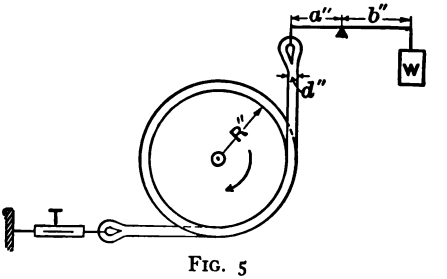


FIG. 5

EXERCISE 6. (a) What must be the length of the brake arm, $l$ (Fig. 4), in inches, so that the "effective pressure" will be numerically equal to the brake H.P. per 1000 r.p.m.?

(b) A prony brake the length of whose arm is that calculated in (a) gives an " effective pressure " of 200 pounds when the shaft makes 500 r.p.m. What is the brake H.P.?

EXERCISE 7. (a) Find the H.P. absorbed in the brake shown in Fig. 5; here the energy is changed to heat by the friction of a rope making one and a quarter turns about the wheel.

(b) Why should the shaft rotate as shown?

## Transmission Dynamometers

The transmission dynamometer shown in Fig. 6 consists of three gear wheels $A$, $B$, and $C$. The wheels $A$ and $C$ are fastened to fixed shafts at $A$ and $C$. These shafts transmit the energy to and from the dynamometer. The wheel $B$ is fastened only to the T-shaped arm at $B$. This arm itself turns freely about $C$ as a center; it does not rotate with the shaft driven by wheel $C$.

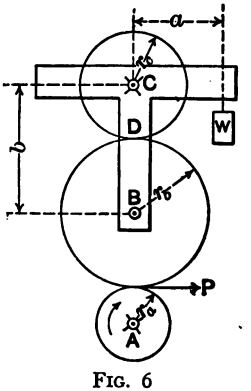

FIG. 6

If the gear $A$ drives in the direction shown and the weight $W$ is removed or is too small, the gear $B$ will not transmit motion to $C$ but will simply be displaced towards the right, moving about $D$ as a center. If, however, $W$ is heavy enough to keep the T-shaped arm in the position shown, then the gear $B$ rotates about $B$ and transmits motion and energy to the shaft $C$. If the force transmitted at the pitch circle from $A$ to $B$ is $P$ pounds, then the force at $B$ required to hold gear $B$ in the position shown in Fig. 6 will be $2P$ pounds (see Ex. 8). The

weight, $W$, necessary to furnish this force at $B$ is $W = \frac{2Pb}{a}$ pounds and the work per revolution of the driver will be

$$2\pi r_a P = \frac{\pi r_a W a}{b} \text{ foot-pounds.}$$

Thus by adjusting $W$ and counting the revolutions per minute at the driver the horse-power transmitted can be measured.

EXERCISE 8. Show gear $B$ and the arm in Fig. 6 as free bodies and from these deduce the relations between the forces used in the above discussion.

EXERCISE 9. Calculate the work per minute delivered by gear $C$, Fig. 6, and compare the result with the work supplied per minute to gear $A$, which makes $N$ r.p.m.

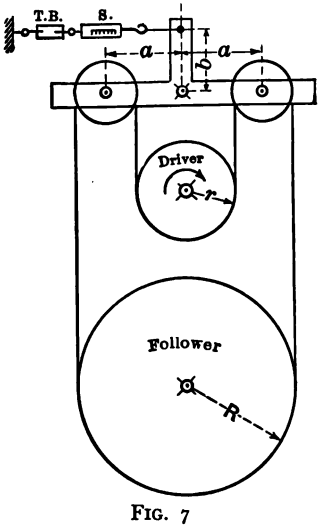

FIG. 7

EXERCISE 10. Fig. 7 shows a transmission dynamometer in which a belt transmits the energy. The crosses (×) indicate fixed centers. The turnbuckle, $T.B.$, must be adjusted so that the T-shaped arm remains in the position shown in the sketch. Note that the tension in the belt on one side of the driving pulley is greater than the tension on the other side. Assuming the reading of the spring balance to be $S$ pounds, when the dynamometer is transmitting power and the driver makes $N$ r.p.m., calculate the H.P. transmitted by considering the work done at

(a) the driver,

(b) the follower.

In the **Briggs dynamometer** the belt transmitting the energy to be measured passes over two equal pulleys, Fig. 8, and over two idle pulleys attached to a frame, $A$,

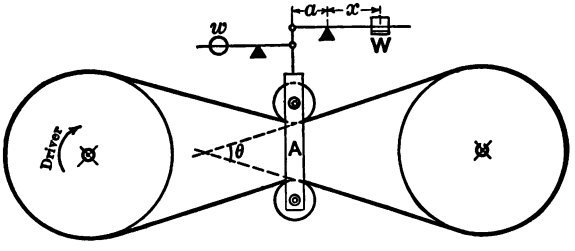


FIG. 8

so arranged as to allow only a slight vertical motion. The weight of this frame and its pulleys is balanced by a counter-weight $w$. When the belt transmits no energy the frame, $A$, takes a symmetrical position. As soon as energy is transmitted the frame tends to move downward owing to the increased tension on the lower side of the belt. Any vertical displacement of the frame, $A$, is prevented by adjusting the weight, $W$.

Fig. 9 shows the frame, $A$, as a "free body" and from it the difference in the tensions $(T_2 - T_1)$ can be obtained in terms of the angle $\theta$ and $W$ by placing the sum of the vertical components of the forces equal to zero. Thus

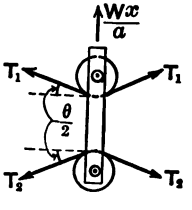


FIG. 9

$$2\,T_1 \sin\frac{\theta}{2} - 2\,T_2 \sin\frac{\theta}{2} + \frac{Wx}{a} = 0,$$

whence

$$T_2 - T_1 = \frac{Wx}{2\,a \sin\frac{\theta}{2}}.$$

EXERCISE 11. If the difference in the tensions in the belt of a Briggs dynamometer is to equal twice the force required to hold the idlers in the central position, what should be the angle between the two parts of the belt leaving the driver?

EXERCISE 12. Assuming the difference in the tensions in the two sides of a belt to be $(T_2 - T_1)$ pounds, what H.P. does the belt transmit if its linear velocity is $V$ feet per minute?

The **Webb floating dynamometer** is often used to measure the torque transmitted by an electric motor to the driven machinery or the torque necessary to drive a dynamo. This dynamometer consists of a tank in which floats a "negative" tank or float allowing just sufficient play so that no contact occurs on slight displacement. Upon this float the dynamo or motor is mounted and by adjusting weights within it the float is leveled. If now a torque is applied to the armature and transmitted by the magnetic field to the magnets of the dynamo the level of the float is disturbed. This tipping of the float is prevented or corrected by shifting a known weight thru a measured distance in a horizontal direction and in a plane at right angles to the axis of the applied torque.

By this means the tilting torque is measured. For our purpose the float and attached dynamo may be considered as a free body upon which the upward buoyant force due to the displaced water balances the downward weight of the float, the dynamo, the weights used for leveling the float, and the shifting weight in its first position, i.e., its position when the float is level and no torque is transmitted to it. Shifting a weight of $w$ pounds a distance of $d$ feet in order to restore the level position when

a torque is transmitted means that the applied torque is $wd$ pound-feet. That this is so can be demonstrated as follows. Consider the float in equilibrium when no torque is transmitted, as shown in Fig. 10. Here the arrow marked $W$ represents the resultant of all weights **except the shifting weight** which is represented by $w$. These forces are balanced by the buoyant force $B = W + w$, no torque being applied to the float thru the shaft. We thus have

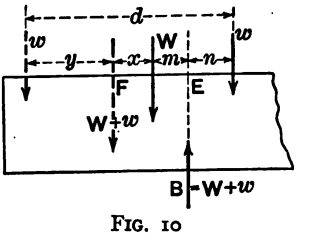

FIG. 10

$$Wm = wn. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If now $w$ is shifted to the position indicated by the dotted arrow $w$, $d$ feet from its first position, the resultant of all the weights will move from $E$ to $F$ and

$$wy = Wx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Under these conditions an unbalanced couple formed by the buoyant force, $B = W + w$, and the new (dotted) resultant thru $F$, $W + w$, acts upon the float; this couple is necessary to balance the torque to be measured, which is applied to the float thru the shaft of the motor or dynamo.

This torque is $(W + w)(x + m)$ and by means of relations (1) and (2) this reduces to $wd$.

EXERCISE 13. Complete the above demonstration. Why does not the buoyant force $B$ shift its line of action when a torque is applied and $w$ is shifted?

EXERCISE 14. (a) Is the shaft transmitting the torque to the dynamometer shown in Fig. 10 normal or parallel to the plane of the paper?

(b) A weight of 10 pounds must be shifted 2 feet 5 inches to level the float of a Webb floating dynamometer when a motor fastened to the float drives a direct-connected centrifugal pump at 300 r.p.m. What H.P. is required to run this pump?

EXERCISE 15. If the scale used to determine the position of the sliding weight on a Webb floating dynamometer is graduated in feet and decimals of a foot, what should be the weight of the sliding weight so that every hundredth of a foot thru which this weight is moved represents a thousandth of a H.P. per hundred r.p.m.?

## SECTION III

### WORK DONE BY VARIABLE FORCES

From the definition of work it follows that the work done by a variable force, $F$, whose line of action coincides with the differential displacement, $ds$, is

$$\int_{s=s_1}^{s=s_2} F\,ds.$$

If $F$ can be expressed as a function of $s$ then the indicated integration can be performed and the total work done can be calculated.

**Work Done by the Force of Gravitation.** — Whenever a body is lifted work is done against the force of gravitation. It is usually assumed that this force is constant and for practical engineering problems, where the height to which bodies are lifted is small compared to the diameter of the earth, this is sufficiently exact.

If, however, the exact work done against gravity is required, the attraction of gravity must be considered as variable. The law of variation is expressed as follows: The attraction of two spheres varies directly as the prod-

uct of their masses and inversely as the square of the distance between their centers.

Thus, to calculate the work done in lifting a sphere whose weight at the earth's surface is $W$ pounds to a height of $h$ feet above the surface, Fig. 11, consider the mass at a height of $x$ feet above the surface. The attraction of the earth is then $F$ pounds where

$$F \propto \frac{Mm}{(R+x)^2},$$

and where $M$ is the mass of the earth, $m$ the mass of the sphere, and $R$ the radius of the earth expressed in feet.

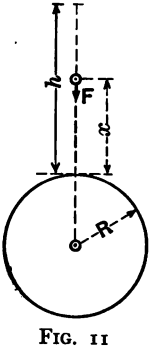

FIG. 11

Thus,
$$F = \frac{c}{(R+x)^2},$$

where $c$ is some constant. To find $c$ note that $F$ at the earth's surface (where $x = 0$) is $W$, or

$$W = \frac{c}{R^2},$$

whence
$$c = WR^2$$

and
$$F = \frac{WR^2}{(R+x)^2}.$$

$$\therefore \text{ The work done} = \int_{x=0}^{x=h} -F\,dx = \int_{x=0}^{x=h} -\frac{WR^2}{(R+x)^2}\,dx$$

$$= +WR^2\left[\frac{1}{R+x}\right]_{x=0}^{x=h} = -\frac{WRh}{R+h}.$$

EXERCISE 16. Solve the above problem when $x$ is measured from the center of the earth.

EXERCISE 17. If $h$ is very small compared to $R$, what value does $\frac{WRh}{R+h}$ approximate?

EXERCISE 18. What work is done in lifting a 200-pound weight from the surface of the earth (radius 4000 miles)

(a) to a height of 1000 feet above the surface?

(b) to an infinite distance from the center of the earth?

**Work Done in Deforming a Spring.** — A spring follows Hooke's Law provided the stress is kept within the elastic limit. According to Hooke's Law the deformation is proportional to the force applied.

The **strength of a spring** is expressed by the force (in pounds) required to increase or decrease its original (unstressed) length one foot.

Thus a spring whose strength is 10 pounds per foot would require a force of 10 pounds to stretch it 1 foot, or 30 pounds to stretch it 3 feet, provided the elastic limit is not exceeded by this deformation.

As an illustration consider the work done in stretching a spring, whose natural length is 10 feet and whose strength is 10 pounds per foot, to a length of 13 feet. In Fig. 12 (a) let $x$ represent a stretched length of the spring; its deformation is then $(x - 10)$ feet and the force, $F$, required to keep it stretched to this length is $(x - 10)$ $(10)$ pounds.

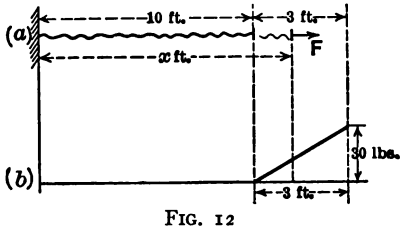

FIG. 12

The work done in stretching the spring is

$$\int_{x=10}^{x=13} F\,dx = \int_{x=10}^{x=13} (x - 10)\,(10)\,dx = 45 \text{ foot-pounds.}$$

As the variable force $F$ is a **linear function of the elongation** the work can also be found without integra-

tion by considering the **average value of the variable force.**

Thus the force applied at the beginning of the elongation is 0 pounds and at the end 30 pounds, and as it **increases directly with the elongation** the average tension is $\frac{0+30}{2}$ or 15 pounds, and the work done equals (average force) (displacement) = (15) (3) = 45 foot-pounds.

Fig. 12 (b) illustrates this graphically, the triangle whose base is 3 feet and whose altitude is 30 pounds being the **work diagram;** the ordinate at any point represents to scale the tension of the spring or the force applied to produce the corresponding elongation. The area of this work diagram $\left(\frac{3 \times 30}{2} = 45\right)$ represents the work done in stretching the spring.

EXERCISE 19. What must be the strength of a spring 10 inches long and capable of storing 1000 foot-pounds of energy with a compression of 2 inches?

EXERCISE 20. A spring whose natural length is 3 feet and whose strength is 60 pounds per foot is to be stretched from a length of 2.5 feet to a length of 4.25 feet. Calculate the work done

(a) by calculus;

(b) by means of the average force.

(c) Draw the work diagram for this problem and from it obtain the work done.

**Work Done by the Expansion of a Perfect Gas at Constant Temperature** (Isothermal Expansion).

In Fig. 13 (a) let (1) represent the position of a piston enclosing a certain mass of gas, the volume and pressure of this gas being $v_1$ and $p_1$ respectively. Similarly $v_2$ and

$p_2$ are the corresponding values after expansion to the position shown by the piston (2). In Fig. 13 (b) these conditions are represented graphically. The curve joining points 1 and 2 is the graphical representation of **Boyle's Law**, the **law governing the isothermal expansion of a gas.** This law expressed analytically is $pv = c$, a constant, and the curve representing it is therefore an equilateral hyperbola referred to its asymptotes as axes.

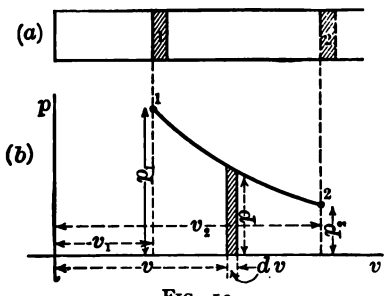

FIG. 13

The work done by an expanding gas can best be expressed in the form $(v_2 - v_1)\ p$, provided the pressure, $p$, is constant. If $p$ varies then the change in volume $(v_2 - v_1)$ must be considered infinitesimal and expressed by $dv$. During this change in volume $p$ is constant and the work done is $p\ dv$,

so that the work done $= \int_{v_1}^{v_2} p\ dv.$

In the example illustrated in Fig. 13,

$$pv = c, \qquad \therefore \quad p = \frac{c}{v},$$

or work done $= \int_{v_1}^{v_2} \frac{c}{v} dv = c\left[\log_e v\right]_{v_1}^{v_2} = c \log_e \frac{v_2}{v_1}.$

To find $c$ note that $p = p_1$ when $v = v_1$, so that $p_1 v_1 = c$. Thus the work of isothermal expansion from a volume

of $v_1$ cubic feet and pressure of $p_1$ pounds per square foot to a volume of $v_2$ cubic feet is

$$p_1 v_1 \log_e \frac{v_2}{v_1} \text{ foot-pounds.}$$

Note carefully that this expression represents the area of the work diagram, in Fig. 13 (b), bounded by the hyperbola, the ordinates $p_1$ and $p_2$, and the axis of $v$.

**Isothermal Expansion; the Hyperbola.** — The following is a simple construction for plotting the hyperbola as used in work diagrams involving the isothermal expansion of gases.

As in Fig. 14 assume axes of zero volume and zero pressure; also some point $(v_1, p_1)$ representing known conditions of volume and pressure.

Thru this point draw a line parallel to the $v$-axis, then thru the origin any oblique lines. The figure shows how points on the curve are then located.

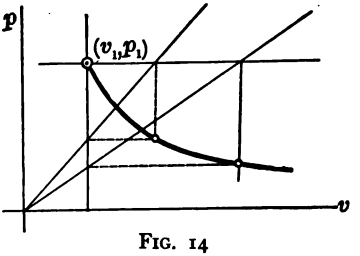


FIG. 14

EXERCISE 21. Show that the construction illustrated in Fig. 14 produces a curve whose equation is $pv$ = constant $= p_1 v_1$. Extend the construction of the curve in Fig. 14 above the point $(v_1, p_1)$.

**Theoretical Indicated Horse-Power of the Steam Engine.** — In the steam engine, steam is admitted to the cylinder at a constant pressure for a part of the stroke. The steam is then cut off and the steam already admitted is allowed to expand, the pressure in the meantime falling from $p_i$, the initial pressure, to some pressure usually

greater than the back pressure, $p_b$. This back pressure is due to the atmospheric pressure and the resistances encountered by the exhaust steam in leaving the cylinder. These conditions are represented graphically in the work diagram in Fig. 15, the shaded area of which represents the work done on one side of the piston during two strokes.

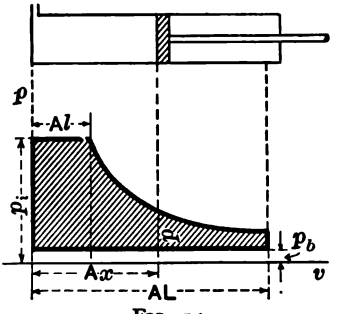

FIG. 15

This theoretical work is readily found by subdividing the work diagram into two rectangles and the area under the hyperbola. Thus the work done equals

$$Alp_i + \int_{x=l}^{x=L} pA\,dx - ALp_b.$$

EXERCISE 22. Show that the indicated H.P. reduces to

$$\frac{2ALN\left\{\frac{p_i}{r}(1+\log_e r) - p_b\right\}}{33{,}000}$$

for a double acting engine for which the **ratio of expansion** $\frac{L}{l}$ is $r$. In what units must the various quantities in the above formula be expressed?

EXERCISE 23. What is the mean effective pressure for the work diagram in Fig. 15?

EXERCISE 24. The diameter of the piston of a steam engine is 40 inches, the stroke is 5 feet, the number of revolutions 50 per minute. If the steam enters the cylinder at 80 pounds absolute, is cut off at one-third of the stroke and the

back pressure is 3 pounds absolute, find the I.H.P. of the engine.

EXERCISE 25. A triple-expansion steam engine has three cylinders 26.5, 36.5, and 54 inches in diameter; the stroke of each cylinder is 5 feet. The steam pressure in the main pipe is 150 pounds per square inch; in the receiver between high and intermediate cylinders, 40 pounds; between intermediate and low, 5 pounds; and a vacuum of 27 inches is maintained in the condenser. The mean effective pressures in the cylinders are: high, 54; intermediate, 23.5; and low, 12.5 pounds per square inch. Under the above conditions the engine ran at 65 r.p.m. Find the I.H.P.

EXERCISE 26. A boiler supplies a steam engine with 100 cubic feet of steam per minute at an average pressure of 50 pounds per square inch. What is the I.H.P. of the engine?

In calculating the I.H.P. of a steam engine as outlined on page 18 it was assumed that the piston came in contact with the cylinder head at the end of its stroke. This of course is impractical. A certain "**clearance**" must always be allowed between the cylinder head and the piston at the end of its stroke and the steam passages from the valves to the cylinder add to this clearance volume. The volume due to the clearance must be filled with steam before the piston moves. Instead of using "live steam" for this purpose the exhaust port is closed before the piston reaches the end of its stroke and the imprisoned steam is compressed from the back pressure to some higher pressure usually below the initial steam pressure.

**The theoretical indicated horse-power of a steam engine,** taking into account **clearance and compression, may** be computed as follows:

Fig. 16 (a) shows the work diagram for the forward stroke of the piston (to the right) and the shaded area represents the work done by the steam upon the piston. Fig. 16 (b) shows the work diagram for the return stroke of the piston (to the left) and the shaded area represents the work done by the piston upon the steam first in exhausting some of it from the cylinder and then in compressing the remainder into the clearance volume. This clearance volume is expressed as a fraction of the stroke; thus $c$ (in Fig. 16) represents the length of a cylinder, having the same area as the piston, whose volume equals the clearance volume. In Fig. 16

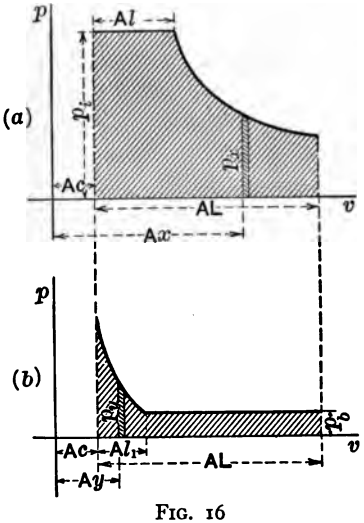


FIG. 16

$A$ = the area of piston.
$L$ = the length of the stroke.
$l$ = the distance the piston travels before "cut-off" occurs.
$l_1$ = the distance the piston travels during compression.
$c$ = the "clearance" as explained above.
$p_i$ = the initial steam pressure.
$p_b$ = the back pressure.

From Fig. 16 (a) the work done on the piston during one stroke is

$$W = Alp_i + \int_{x=(c+l)}^{x=(c+L)} p_x(A\,dx) = Ap_i\left\{l + (l+c)\log\left(\frac{L+c}{l+c}\right)\right\},$$

for according to the law $pv = c$ we have

$$p_x(Ax) = p_iA(l + c).$$

From Fig. 16 (b) the work done by the piston during the return stroke is

$$W' = A(L - l_1)\,p_b + \int_{y=c}^{y=c+l_1} p_y\,(A\,dy)$$

$$= Ap_b\left\{L - l_1 + (l_1 + c)\log\left(\frac{l_1 + c}{c}\right)\right\}.$$

The effective work per stroke is then $W - W'$.

EXERCISE 27. Complete the calculations in the above indicated steps.

EXERCISE 28. What is the mean effective steam pressure according to the data assumed for Fig. 16? In what units must each quantity used be expressed?

EXERCISE 29. Referring to Fig. 16 calculate (a) the pressure in the cylinder at the end of the expansion; (b) the pressure due to compression when admission occurs; (c) assuming the back pressure, the clearance, and the point at which compression commences, construct the compression curve.

**Work Done by the Adiabatic Expansion of Air.** — Air expands adiabatically when its entropy remains constant. In this case the gas neither receives nor emits heat. The law of adiabatic expansion is $pv^n$ = constant where $n = 1.4$. In practice the conditions for adiabatic change are seldom realized and then $n$ takes values lying between 1 and 1.4.

The curves representing the equation $pv^n$ = constant may readily be constructed in the following manner.*

* E. Brauer; Z. d. V. d. I., p. 433, 1885.

In the equation $(1 + \tan \beta) = (1 + \tan \alpha)^n$ assume $\alpha$ and compute $\beta$. Then as in Fig. 17 lay down the angles $\alpha$ and $\beta$ from the axes of $v$ and $p$ respectively as shown; now starting from any known point $(v_1, p_1)$ on the required curve draw lines at 45° as indicated to locate another point on the curve. Starting with this point proceed to find the next point, etc.

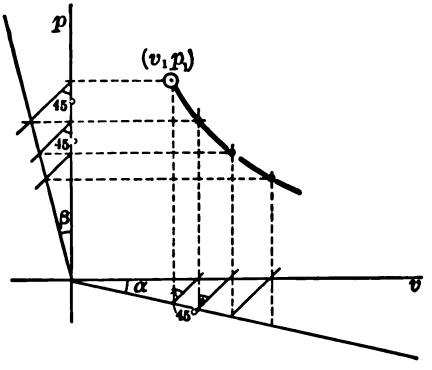


FIG. 17

EXERCISE 30. Prove that the above construction gives a curve whose equation is $pv^n = \text{constant} = p_1 v_1^n$.

EXERCISE 31. (a) How would you cause the constructed points on the curve in Fig. 17 to lie closer together?

(b) Prove the above construction for $n = 1$. What curve does this produce?

EXERCISE 32. Show that the work done in compressing a gas according to the law $pv^n$ = a constant from a volume and pressure of $v_1$ and $p_1$ to a smaller volume $v_2$ and higher pressure $p_2$ is

$$\frac{p_1 v_1}{n-1}\left\{\left(\frac{v_1}{v_2}\right)^{n-1} - 1\right\}, \quad \text{or} \quad \frac{p_1 v_1}{n-1}\left\{\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} - 1\right\}.$$

**To Find the Work Done by Varying Forces when the Law of Variation Cannot be Expressed Analytically.** — When the law governing the relation between force and displacement cannot be expressed analytically then the calculus cannot be used to compute the work done. In

cases of this kind the law of variation is either represented graphically by means of a curve (obtained experimentally as on the indicator card), or certain instantaneous values of force and of displacement are known.

The problem thus reduces to that of finding the area bounded by one force-displacement curve, two ordinates, and the axis of displacement.

The following rules, given in their order of approximation, are used to estimate this area. In each of these rules the area is assumed to be bounded by the given curve and three straight lines, two of which are perpendicular to the third.

These rules are only applicable to such intervals of curves in which no sudden changes in rate of curvature occur.

**A. The Trapezoidal Rule.** — Fig. 18 represents the figure whose area is sought. Divide the horizontal line into any number of equal parts, of length $l$, and erect ordinates at the points of division. Let the lengths of these ordinates be $y_0$, $y_1$, $y_2$, etc.

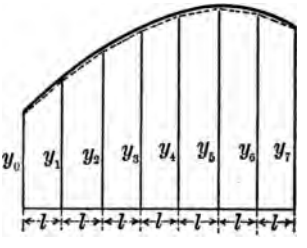

FIG. 18

Then assuming the area to be composed of trapezoids of common altitude $l$ (as shown by the dotted lines),

$$\text{Area} = \frac{l}{2}(y_0 + y_1) + \frac{l}{2}(y_1 + y_2) + \ldots$$

$$= \frac{l}{2}(y_0 + 2\,y_1 + 2\,y_2 + \ldots y_n)$$

$$= l\left(\frac{y_0}{2} + y_1 + y_2 + \ldots y_{n-1} + \frac{y_n}{2}\right).$$

In case the curve is not given, only certain points upon it being fixed as determined by their ordinates, then the assumption that the curve consists of straight lines joining the ends of the ordinates is probably as true as any other assumption and the above formula gives as accurate an answer as can be expected. If, however, the curve is known, better results are obtained by means of

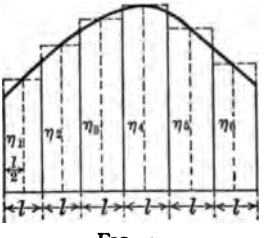

FIG. 19

**B. The Rectangular Rule.** — Under this rule the area is conceived as composed of rectangles, as indicated in Fig. 19. The assumption here made consists in regarding the area added to the actual strip as equal to the area subtracted from it in transforming the actual strips into the rectangles. Thus the

$$\text{Area} = l\,(\eta_1 + \eta_2 + \eta_3 + \ldots \eta_n).$$

**C. Simpson's Rule.** — In applying Simpson's Rule the area must be divided into an **even number of strips** having a width $l$.

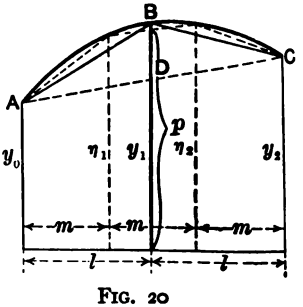

FIG. 20

In Fig. 20 consider two adjacent strips, the ordinates being $y_0$, $y_1$, and $y_2$.

Under Rule A the area would be

$$A = \frac{l}{2}(y_0 + y_1) + \frac{l}{2}(y_1 + y_2).$$

As a closer approximation divide the width, $2\,l$, of these strips into 3 equal parts each equal to $m$. We then have

$$A = \frac{m}{2}(y_0 + \eta_1) + \frac{m}{2}(\eta_1 + \eta_2) + \frac{m}{2}(\eta_2 + y_2)$$

$$= \frac{m}{2}(y_0 + 2\eta_1 + 2\eta_2 + y_2);$$

but as $3m = 2l$, and $\eta_1 + \eta_2 = 2p$,

$$A = \frac{l}{3}(y_0 + 4p + y_2).$$

If now the curve is concave downward, as in Fig. 20, the value of $A$ just obtained is too small and thus by substituting for $p$ the length $y_1$, which is slightly longer, we obtain

$$A = \frac{l}{3}(y_0 + 4y_1 + y_2),$$

as a closer approximation of the actual area to be found.

It readily follows that for $2n$ intervals of width $l$ Simpson's Rule becomes

$$A = \frac{l}{3}[y_0 + y_{2n} + 2(y_2 + y_4 + \cdots y_{2n-2}) + 4(y_1 + y_3 + \cdots y_{2n-1})].$$

EXERCISE 33. Prove the last statement.

EXERCISE 34. Prove Simpson's Rule for a curve which is concave upward.

Another proof of Simpson's Rule is based upon the assumption that the curve passing through the points $A$, $B$, and $C$, Fig. 20, is a parabolic arc whose tangent at $B$ is parallel to $AC$.

If, then, we remember that the area of the parabolic segment $ABC$ equals two-thirds of $AC$ times the perpendicular distance from $B$ to $AC$ or its equivalent

$$\frac{2}{3}(2l)(BD) = \frac{4}{3}l\left(y_1 - \frac{y_0 + y_2}{2}\right),$$

we have the required area

$$A = 2l\left(\frac{y_0 + y_2}{2}\right) + \frac{4l}{3}\left(y_1 - \frac{y_0 + y_2}{2}\right)$$

$$= l(y_0 + y_2) + \frac{2l}{3}(2y_1 - y_0 - y_2)$$

$$= \frac{l}{6}(6y_0 + 6y_2 + 8y_1 - 4y_0 - 4y_2)$$

$$= \frac{l}{3}(y_0 + 4y_1 - y_2).$$

EXERCISE 35. Show that if $y^2 = 2px$ and $y = mx + b$ intersect in $A$ and $C$ then the area between them is two-thirds of $AC$ times the distance between $AC$ and the tangent to the parabola which is parallel to $AC$.

**D. Simpson's "Three-eighths" Rule.** — The above Simpson's Rule, also called Simpson's "one-third" rule, can only be applied when the number of strips is a multiple of two.

Simpson's "three-eighths" rule (applicable only when the number of strips is a multiple of three) follows:

$$A = \frac{3l}{8}(y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \ldots + 3y_{m-1} + y_m).$$

EXERCISE 36. Prove Simpson's "three-eighths" rule by using three intervals and subdividing them into four parts for closer approximation and proceeding in a manner similar to that on page 24.

Numerous other rules for approximating areas such as the above have been devised, notably those of Durand (Engineering News, 31[1], 42, 1894) and Tschebyscheff (Hütte, Vol. II, Edition of 1908).

EXERCISE 37. A variable force is displaced 6 feet and equals 3, 8, 15, 24, 35, 48, and 63 pounds at successive equidistant intervals. Find the work done, by means of

(a) the trapezoidal rule;

(b) Simpson's Rule.

EXERCISE 38. The steam pressures as shown by an indicator for equally spaced points along the 18-inch stroke of the piston of a steam hammer are 45, 50, 50, 45, 30, 25, 20 pounds per square inch. The back pressure is 16 pounds per square inch. If the moving mass of the hammer weighs 640 pounds and the piston area is 25 square inches, what work is stored in the hammer at the end of its stroke?

Areas may also be found by means of **planimeters.**

## SECTION IV

### WORK DONE BY OBLIQUE FORCES

Wherever the line of action of a force does not coincide with the displacement, the effective displacement and the effective force must be used. Thus if a force of $F$ pounds is displaced a distance $ds$, and the angle between their directions is $\theta$, the work done is

$$dW = (F \cos \theta)\, ds.$$

As an **application** let us show that no work is lost between the piston and the crank of an engine by reason of its transmission thru the connecting-rod; of course friction is neglected.

In Fig. 21, assume a constant force $P$ acting upon the piston. Let the length of the connecting-rod be $l$ and of the crank $r$. As a triangle of forces will show, the force transmitted along the connecting-rod is $P \sec \phi$. This force, acting at $A$ along the center line of the connecting-rod, suffers a displacement $ds = r\, d\theta$ at right angles to AC.

The work done during this differential displacement is

$$P \sec \phi \sin (\phi + \theta)\, r\, d\theta.$$

Thus, the work done at the crank pin during one-half a revolution while $\theta$ changes from 0 to $\pi$ radians is

$$W = \int_{\theta=0}^{\theta=\pi} P \sec \phi \sin (\phi + \theta)\, r\, d\theta.$$

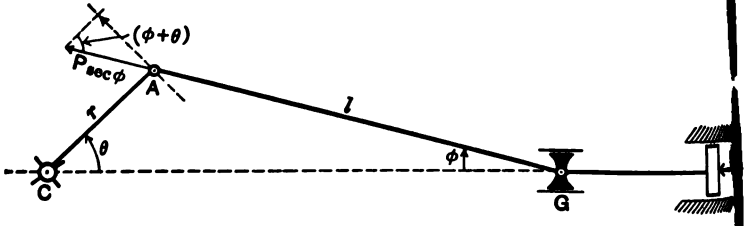


FIG. 21

Here $\phi$ must be expressed in terms of $\theta$ before integration is possible. From $\Delta CGA$ by the "law of sines" we have

$$\sin \phi = \frac{r}{l} \sin \theta,$$

and as $W = Pr \int_0^{\pi} \sec \phi \,(\sin \phi \cos \theta + \cos \phi \sin \theta)\, d\theta$

$$= Pr \int_0^{\pi} (\tan \phi \cos \theta\, d\theta + \sin \theta)\, d\theta,$$

we have

$$W = Pr \int_0^{\pi} \left\{ \sin \theta\, d\theta + \frac{\frac{r}{l} \sin \theta \cos \theta}{\sqrt{1 - \frac{r^2}{l^2} \sin^2 \theta}} d\theta \right\}$$

$$= Pr \left[ -\cos \theta - \frac{l}{r} \sqrt{1 - \frac{r^2}{l^2} \sin^2 \theta} \right]_0^{\pi}$$

$$= 2\, Pr.$$

But $2Pr$ is also the work done at the piston by the constant force $P$ during a stroke whose length is $2r$.

EXERCISE 39. Prove the above for the return stroke.

EXERCISE 40. Show that the work obtained at the crank-pin thru a slotted cross-head, Fig. 22, equals the work done at the piston, friction neglected.

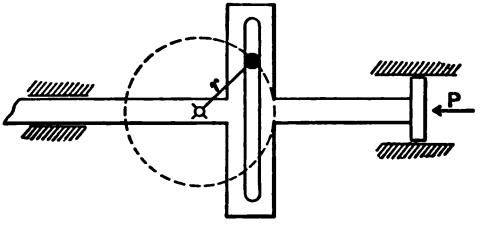


FIG. 22

EXERCISE 41. If the boat, Fig. 23, travels at a velocity of $V$ feet per minute when the wind exerts a force of $P$ pounds in the direction of its relative motion with respect to the boat as indicated by the arrow, find the work done by the wind.

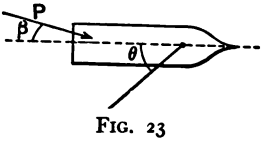


FIG. 23

EXERCISE 42. Assuming $V$ and $P$ constant in Ex. 41, what relation exists between $\theta$ and $\beta$ for maximum work? Express the answer in words.

## SECTION V

### KINETIC ENERGY

Energy is the capacity to do work; it is stored work. Kinetic energy is the energy stored in a moving mass.

The **kinetic energy** of a particle is

$$\frac{mv^2}{2},$$

where $m$ is the mass of the particle and $v$ its velocity.

The **unit of mass** used by English-speaking engineers depends upon the unit of force. The unit of force, the pound, has already been defined as the attraction of the earth for a certain piece of platinum when at a latitude of 45° and at the sea level. For convenience let this certain piece of platinum or any exact copy of it be called a "standard pound."

It can be demonstrated experimentally that if a unit of force were to act freely upon a " standard pound" it would produce an acceleration of 32.17 feet per second per second. Let $m$ represent the units of mass in a "standard pound," then, as

$$\text{force} = (\text{mass})\,(\text{acceleration}),$$

we have,

1 pound = ($m$ units of mass) (32.17 feet per second per second), or $m = \frac{1}{32.17}$ units of mass.

Thus, our "standard pound," which weighs one pound (at a latitude of 45° and at the sea level), contains $\frac{1}{32.17}$ units of mass.

**A unit of mass would therefore weigh 32.17 ($= g_{45}$) pounds at a latitude of 45° and at the sea level.**

The weight of a body (i.e., the force with which the earth attracts it) is usually determined by comparing the earth's attraction for it with the earth's attraction for "standard pounds." We balance the body with standard "weights." Suppose that this comparison is made at a place where the acceleration of gravity is $g$ (other than $g_{45}$), then, as mass remains unchanged with change of locality, the equation

$$\text{force} = (\text{mass})\,(\text{acceleration})$$

shows that each "standard pound," used as above described, is attracted to the earth with a force of

$$W = \left(\frac{1}{32.17} \text{ units of mass}\right) (g \text{ feet per second per second})$$

$$= \frac{g}{g_{45}} \text{ pounds,}$$

and not with a force of one pound as might be supposed.

The true weight of the body is thus not equal to the number of "standard pounds" used to balance it. Its true weight is equal to this number times $\frac{g}{g_{45}}$ pounds.

Similarly any force determined by means of "standard pounds" cannot be measured by the number of "standard pounds" which balance it unless the operation is performed at a latitude of 45° and at the sea level.

In all cases **a force determined in this way will equal in pounds the number of "standard pounds" times** $\frac{g}{g_{45}}$, where $g$ is the acceleration of gravity at the place of experimentation and $g_{45}$ is the acceleration of gravity at a latitude of 45° and at the sea level.

The value of $g$ at any place on the earth's surface may be determined by means of the formula

$$g = 32.17\,(1 - 0.002648 \cos 2\phi) - 0.000003\,h,$$

wherein $\phi$ is the latitude
and $h$ is the altitude, in feet, above the sea level.

To illustrate, a "standard pound" when suspended by a wire at the equator and at the sea level (where $g = 32.08$) produces a tension not of one pound but of $\frac{32.08}{32.17}$ pounds in the wire. Similarly, if 100 "standard

pounds" are required to balance a Prony brake at the equator and at the sea level, the force exerted upon the brake is not 100 pounds but 100 $\left(\frac{32.08}{32.17}\right)$ pounds.

In opposition to the above illustrations consider the measurement of forces by means of a spring balance. Assume **a spring balance** to be **accurately calibrated** by means of "standard pounds" **at a latitude of 45° and at the sea level,** then it **will read in true pounds irrespective of locality.**

In **the determination of mass** matters are quite different. A balance, which determines that 150 "standard pounds" balance a certain iron casting at the equator and at the sea level, shows that the mass of this casting is equivalent to the mass of 150 "standard pounds." Its equilibrium would remain undisturbed if transported to latitude 45° and the sea level, for any change in the earth's attraction would affect both the casting and the "standard pounds" equally. As the mass of each "standard pound" is $\frac{1}{32.17}$ units of mass, the mass of the casting is $\frac{150}{32.17} = \frac{150}{g_{45}}$ units of mass.

Or, from another point of view, the true weight of the casting at the equator and at the sea level is (150) $\left(\frac{32.08}{32.17}\right)$ pounds.

Let $M$ be the mass of this casting, then, as

$$\text{force} = (\text{mass})\,(\text{acceleration})$$

$$(150)\left(\frac{32.08}{32.17}\right) = (M)\,(32.08),$$

or $$M = 150\left(\frac{32.08}{32.17}\right)\left(\frac{1}{32.08}\right)$$

$$= \frac{150}{32.17} \text{ units of mass, as before.}$$

Thus,

**the mass of any body anywhere equals its true weight in pounds divided by the acceleration of gravity at the locality in question.**

Assume now that the same casting is suspended from a spring balance at the equator The spring balance will not indicate 150 pounds as the weight of this casting but $(150)\left(\frac{32.08}{32.17}\right)$ pounds, its true weight. As above shown this true weight must be divided by the acceleration of gravity at the equator (i.e., 32.08) in order to obtain the mass of the casting.

In engineering calculations the distinction between the true weight and the apparent weight is seldom necessary. Thus, the weight of the casting would be stated as 150 pounds even tho it be at the equator and not in latitude 45°. This is equivalent to assuming the acceleration of gravity as constant and equal to 32.17 over the whole surface of the earth. This assumption will never cause an error of more than, say, one-half of one per cent, a negligible error in engineering calculations. We may thus state that,

**for practical purposes the mass of a body is its weight** (as determined by means of either a common or a spring balance) **divided by 32.17.**

The **unit of velocity** is the foot per second.

The **unit of kinetic energy** is the foot-pound.

EXERCISE 43. Show that kinetic energy and work have the same dimensions and may therefore be measured in the same units.

The kinetic energy of translation of a body is

$$\frac{M\bar{v}^2}{2},$$

where $M$ is the mass of the body and $\bar{v}$ the velocity of its mass-center.

The kinetic energy of a rotating body is

$$\frac{I\omega^2}{2},$$

where $I$ is the moment of inertia of the body about its axis of rotation and $\omega$ is the angular velocity of the body about the same axis.

The **moment of inertia** of a body is defined by

$$I = \Sigma mr^2,$$

where $m$ is the mass of any particle of the body and $r$ its perpendicular distance from the axis of rotation.

The **unit of moment of inertia** is the moment of inertia of a unit of mass concentrated at a unit distance from the axis of rotation.

It is convenient to express the moment of inertia of a body in terms of the **radius of gyration, $k$**, defined by the equation

$$k = \sqrt{\frac{I}{M}},$$

where $I$ is the moment of inertia of the body about the axis considered and $M$ is the total mass of the body.

The **unit of angular velocity** is the radian per second. Thus $n$ revolutions per minute would equal an angular velocity of $\frac{2\pi n}{60}$ radians per second.

The kinetic energy of a free body both translating and rotating is

$$\frac{M\bar{v}^2}{2} + \frac{I\bar{\omega}^2}{2},$$

where $M$ is the mass of the body, $\bar{v}$ the linear velocity of its mass-center, $I$ the moment of inertia about the axis of rotation passing thru the mass-center, and $\bar{\omega}$ the angular velocity about this axis of rotation.

EXERCISE 44. From the definition of the kinetic energy of a particle show that the kinetic energy of a rotating mass is

$$\frac{Mk^2\omega^2}{2}$$

and define the meaning of these letters.

## SECTION VI

### PRINCIPLE OF WORK

Whenever a **free body** is acted on by forces **the sum of the work due to all of the forces** (taken with their proper signs) **always equals the change in kinetic energy which occurs during the action of these forces.**

This law is a special form of the general law of conservation of energy. The work done by all of the forces acting upon a body must be stored in this body as kinetic energy. As a simple application of this principle let us find the number of revolutions, $x$, that a fly-wheel rotating at the rate of $n$ revolutions per minute would make before coming to rest. Fig. 24 represents the wheel as a free body. The forces acting upon it are its

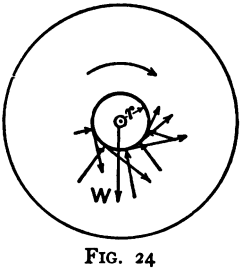


FIG. 24

weight, $W$; the friction forces, totaling to $F$ pounds, which act tangentially at the surface of the journal; and the normal forces $N$.

Of all these forces friction is the only one effectively displaced and thus the only force doing work. This work equals $-2\pi rxF$. The loss in kinetic energy is

$$-\frac{I\left(\frac{2\pi n}{60}\right)^2}{2} = -\frac{\pi^2 n^2 I}{1800},$$

where $I$ is the moment of inertia of the fly-wheel about its axis of rotation.

Thus $$-2\pi rxF = -\frac{\pi^2 n^2 I}{1800},$$

or $$x = \frac{\pi n^2 I}{3600\, rF} \text{ revolutions.}$$

In this discussion $F$ represents the mean frictional force during the interval under discussion. If $F$ is considered constant then the resulting constant angular retardation of the wheel can be found by means of the well-known equations for angular motion under constant angular acceleration, namely:

$$\left.\begin{aligned} \omega &= \alpha t + \omega_0, \\ \theta &= \tfrac{1}{2}\alpha t^2 + \omega_0 t, \\ \omega^2 &= 2\alpha\theta + \omega_0^2, \end{aligned}\right\} \text{ which are analogous to } \left\{\begin{aligned} v &= at + v_0, \\ s &= \tfrac{1}{2} at^2 + v_0 t, \\ v^2 &= 2as + v_0^2, \end{aligned}\right.$$

the equations for linear motion under constant acceleration.

By means of $\omega^2 = 2\alpha\theta + \omega_0^2$, where

$\omega$, the final angular velocity, equals zero,

$\omega_0$, the initial angular velocity, equals $\frac{2\pi n}{60}$ radians per second,

$\theta$, the angular displacement, equals $2\pi x$ radians, we have

$$\alpha = \frac{\omega^2 - \omega_0^2}{2\theta} = \frac{0 - \left(\frac{2\pi n}{60}\right)^2}{4\pi x} = -\frac{n^2\pi}{3600x}$$

$$= -\frac{rF}{I} \text{ radians per second per second.}$$

EXERCISE 45. The rotor of a dynamo has a moment of inertia of 300 units about its axis of rotation and is brought to rest from a speed of 1000 revolutions per minute after making 200 revolutions. What total mean torque, expressed in pound-inches, must have acted upon the rotor during the retardation?

EXERCISE 46. Assuming the torque in Ex. 45 as constant, what kinetic energy did the rotor possess after making 100 revolutions?

As another application of the principle of work consider a cylinder rolling down an inclined plane, Fig. 25. The forces acting upon this cylinder are its weight, $W$; a normal reaction, $N$; and a friction force, $F$, at the point of contact between cylinder and plane. The friction is assumed sufficient to cause pure rolling motion, and as no sliding occurs the friction force, $F$, suffers no effective displacement and does no work.

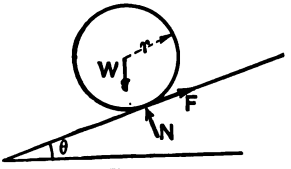

FIG. 25

Thus to find the velocity of the mass-center of the cylinder after it rolls a distance of $s$ feet from rest down the plane we have,

$$\text{work done} = + Ws \sin\theta,$$

$$\text{change in K.E.} = \frac{Wv^2}{2g} + \frac{Wk^2\left(\frac{v}{r}\right)^2}{2g},$$

where $k$ is the radius of gyration of the cylinder about its geometrical axis. Here

$$k^2 = \frac{r^2}{2},$$

thus $$Ws \sin \theta = \frac{Wv^2}{2g} + \frac{Wv^2}{4g} = \frac{Wv^2}{g}\left(\frac{3}{4}\right),$$

and $$v^2 = \frac{4}{3} gs \sin \theta.$$

Note carefully that at any instant $v = \omega r$, where $v$ is the linear velocity of the mass-center and $\omega$ is the corresponding angular velocity about an axis perpendicular to the plane of the paper (in Fig. 25) and thru its mass-center.

EXERCISE 47. How fast and in what direction is a particle on the cylindrical surface and vertically above the point of contact in Fig. 25 moving at any instant?

EXERCISE 48. Compare the velocities of a sphere and a cylinder having equal masses and equal radii after they roll from rest down equal distances on an inclined plane.

EXERCISE 49. Compare the K.Es. of the sphere and cylinder in Ex. 48.

EXERCISE 50. Referring to Fig. 2, what velocity would $W_2$ attain after $W_1$, moving with an initial velocity of $v_0$ feet per second, rises $s$ feet? Consider the wheel and the axle as cylinders of masses $M$ and $m$ respectively and neglect the mass of the shaft and all frictions.

EXERCISE 51. Referring to Fig. 26 which represents a machine in which the force applied to a hand-wheel is transmitted by means of bevel gears, a worm and wheel, and spur gears to a drum 6 inches in radius,

(a) what displacement of $W$ results per turn of the hand-wheel? (P.D. = pitch diameter.)

(b) what weight $W$ can be raised with constant velocity

by a force of 30 pounds applied tangentially to the handle of the hand-wheel?

(c) If the efficiency of each pair of spur and bevel gears is 90% and that of the worm and wheel is 40% including the friction of the corresponding bearings, what weight $W$ can be raised by means of the force applied in (b)?

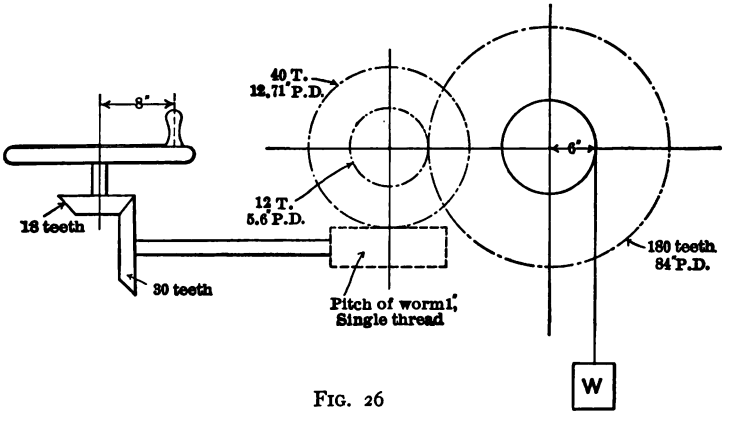


FIG. 26

**Tractive Force of a Locomotive.** In all applications of the principle of work it is exceedingly important to fix the attention upon some part (or connected parts, or even the whole) of the mechanism in question and to represent it as a **free body showing all the forces acting upon it.**

In the case of a locomotive, when the tractive force is required, the whole mechanism may be considered as a free body (represented diagrammatically in Fig. 27). Here the forces acting are the weight, $W$; the normal reactions, $N$ and $N_1$, on the wheels; the mean effective steam force, $P$, acting both upon the piston and the cylinder head; the tractive force, $T$; and the adhesion, $F$, between the drivers and the rails. This adhesion is a friction

force which prevents the drivers from slipping upon the rails. This adhesion is not displaced, in the usual sense, and so does no work as the locomotive advances, provided slipping does not occur. All other frictional forces are purposely omitted; these will be studied later.

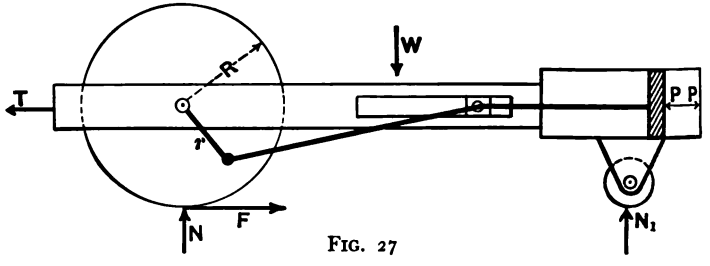


FIG. 27

To avoid complications consider the locomotive running at a constant velocity so that its kinetic energy does not change. Under this condition the work done by all the forces acting upon it as a free body must sum to zero.

As a convenient displacement consider one stroke of the piston or one-half a revolution of the drivers; then the force, $P$, on the cylinder-head does work to the extent of $(P\pi R)$,

the force, $P$, on the piston does work to the extent of $-P(\pi R - 2r)$,

the force, $T$, on the frame does work to the extent of $-T(\pi R)$,

and all other forces do no work.

Thus $$-T(\pi R) - P(\pi R - 2r) + P(\pi R) = 0,$$

or $$T = \frac{2rP}{\pi R},$$

when one cylinder only is considered.

In any application of the principle of work, such as the above, numerous forces other than those shown in Fig. 27 act thruout the mechanism. For instance, a force is transmitted to the frame at the bearings carrying the journals of the drivers; this force is not shown in Fig. 27 for the frame is not considered the free body, since both frame and drivers form the unit considered. Also, any work done by this force transmitted from journals to frame would in the present discussion be counterbalanced by the equal work done by its equal reaction, the force transmitted from frame to journals.

It is also important to note that the value of $T$ found above is the **mean tractive force**, for during the displacement considered various parts of the mechanism (such as the piston, connecting-rod, etc.) change their velocities and thus their kinetic energy. Any energy absorbed during the cycle considered above is returned before the end of the cycle is reached so that a true mean value is obtained. The actual variation in the velocities or the accelerations which occur will be studied later.

EXERCISE 52. Solve the above problem when the relative motion of the piston is forward.

EXERCISE 53. Solve the above problem by considering as free bodies first the rotating and oscillating parts of the locomotive (thus finding the force exerted by the frame upon the journals of the driving-wheel axle; and then the frame itself. The crank is to be taken in the position shown in Fig. 27.

EXERCISE 54. Same as Ex. 53, but the piston is to be considered during its forward stroke.

EXERCISE 55. What should be the mean effective pressure in the two cylinders of a single expansion locomotive having

5-foot drivers, a stroke of 2 feet, a piston area of 100 square inches, and developing a tractive force of 2000 pounds?

EXERCISE 56. (a) Why should passenger engines have drivers of larger diameter than freight engines?

(b) What is the relation between the tractive force and the adhesion?

EXERCISE 57. Express the tractive force of a locomotive in terms of $d$, the diameter of the cylinder in inches; $D$, the diameter of the driving-wheel in inches; $S$, the stroke in inches; and $p$, the m.e.p. in pounds per square inch.

EXERCISE 58. Show that the resistance which a train weighing $W$ tons offers *to acceleration* while changing its velocity from $V_1$ to $V_2$ miles per hour during a run of one mile is $0.01265\,(V_2^2 - V_1^2)\,W$.

EXERCISE 59. A locomotive running on a level track brings a train weighing 120 tons to a speed of 30 miles per hour in 2 minutes. If the resistance due to friction is 8 pounds per ton, what force does this require at the draw-bar?

EXERCISE 60. (a) What work is done in bringing the train in Ex. 59 to speed?

(b) What H.P. does the locomotive deliver at the draw-bar at the instant that the train attains a speed of 30 miles per hour?

EXERCISE 61. (a) What relation exists between the tractive force, $T$ pounds; the speed, $V$ miles per hour; and the horse-power, H.P., delivered at the draw-bar of a locomotive.

(b) What is the greatest speed a locomotive can attain when developing a given H.P. and overcoming a given resistance?

(c) In coming up to speed what quantities in the answer to (b) vary?

EXERCISE 62. (a) What tractive force is required to haul a train weighing $W$ tons up a grade of $h$ feet per mile at a constant velocity of $V$ miles per hour? (Neglect all friction.)

(b) What H.P. does this require?

**Steam Pressure Required to Start a Locomotive.** — Fig. 28 represents the driving wheel of a locomotive, with the crank vertical and above the axle, as a free body.

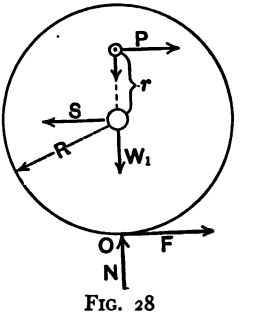

FIG. 28

$P$ is the horizontal component of the force transmitted thru the connecting-rod and equals the total force exerted upon the piston by the steam at the instant considered.

EXERCISE 63. Prove that the last statement is correct.

$S$ is the force transmitted by the frame of the locomotive to the journals of the driver.

Considering the driver in equilibrium and taking moments about $O$ (why about $O$?),

we have $$P(r + R) = SR \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The only horizontal forces (neglecting friction) acting upon the frame will be the force, $P$, exerted by the steam on the cylinder head and acting towards the left, the tractive force also acting towards the left, and the force, $S$, towards the right. (Sketch the frame as a free body.) Thus, for equilibrium we have

$$T + P = S. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2) by elimination of $S$ we obtain

$$P = \frac{RT}{r}.$$

EXERCISE 64. What force exerted on the piston by the steam is required to start the locomotive when the crank is vertical and below the axle of the driver?

EXERCISE 65. (a) How great a force at the draw-bar will cause slipping of the drivers in terms of a given adhesion, $F$, and an unlimited steam pressure?

(b) Why is the crank in the above discussion assumed vertical? Does this detract from the generality of the solution?

# CHAPTER II

## REVIEW OF THE PRINCIPLES OF CO-PLANAR STATICS *

STATICS is that branch of mechanics in which the relations existing between forces acting upon a body at rest, or moving without acceleration, are discussed.

### SECTION VII

### RESULTANTS OF COPLANAR FORCES

### A. Concurrent Forces

**Graphically.** — Let the given forces be represented in Fig. 29 (a).

Fig. 29 (b) shows the given forces placed end to end, forming the **polygon of the forces.** The line *ae* (note carefully its direction) is the resultant of the given forces; a force parallel and equal to it passing thru the common point of intersection of the given forces in Fig. 29 (a) would correctly replace the given forces.

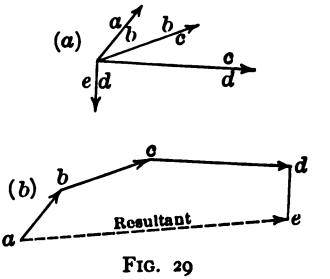

FIG. 29

In constructions such as the above, two cases may occur:

* For a fuller discussion and the necessary proofs see **Martin**, "Text-Book of Mechanics," Vol. I.

(1) $a$ and $e$ in the polygon of forces may not coincide, in which case $ae$ is the resultant of the given forces.

(2) $a$ and $e$ may coincide; then the resultant of the given forces is zero, and the forces are in equilibrium.

**Analytically.** — The forces shown in Fig. 29 (a) can be resolved into components along any two directions at right angles to each other; if the sums of the components along these lines are represented by $\Sigma X$ and $\Sigma Y$ respectively then the resultant, $R$, is

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2},$$

and its direction referred to the axis of $X$ is

$$\tan^{-1}\frac{\Sigma Y}{\Sigma X}.$$

Here we have for **equilibrium** $\Sigma X = 0$ and $\Sigma Y = 0$.

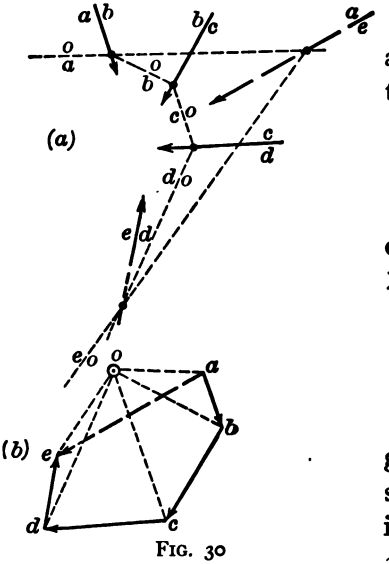


FIG. 30

### B. Any Given Set of Forces

**Graphically.**—Let the given forces be represented by the full lines in Fig. 30 (a); then Fig. 30 (b) shows the corresponding polygon of forces. Here $ae$ represents the resultant in magnitude and direction. To find its position select any point $o$, Fig. 30 (b), as pole and draw the rays $oa$, $ob$, $oc$, etc. Then in Fig. 30 (a) construct the **funicular polygon** $oa$, $ob$, $oc$, $od$, and $oe$, these lines being respectively parallel to the rays of like name. Prolong

first and the last side of the funicular polygon (*oa* and until they intersect and thru this intersection draw the resultant.

Figs. 30 (a) and 30 (b) are called reciprocal figures; lines parallel to any three intersecting lines, such as *ab*, *ob*, or *oc*, *cd*, *od*, etc., in the first figure will always m a triangle in the second figure. Also, note that in . 30 (a)

*oa* and *ob* intersect on *ab*,
*oc* and *od* intersect on *cd*,
*oa* and *oe* intersect on *ae*.

In constructions such as the above, three cases may se:

(1) *a* and *e* in the polygon of forces, Fig. 30 (b), may t coincide; then *ae* is the resultant of the given set of ces located as in Fig. 30 (a).

(2) *a* and *e* in Fig. 30 (b) may coincide; then *oa* and *oe* Fig. 30 (a) must be parallel. In this case the resultant ce *ae* is zero; if *oa* and *oe* in Fig. 30 (a) do not coincide, **esultant couple,** whose forces are *ao* and *oe* in the poly- n of forces, Fig. 30 (*b*), and whose arm is the perpen- ular distance between the sides *oa* and *oe* in the nicular polygon, Fig. 30 (*a*), **is the equivalent of the en forces.**

EXERCISE 66. Change the force *de* in Fig. 30, so that set of given forces has a couple as resultant. Indicate arly the distance which must be measured in order to com- te the moment of the resultant couple; also indicate the gnitude of the forces forming this couple.

(3) *a* and *e* in Fig. 30 (b) may coincide and *oa* and *oe* Fig. 30 (a) may also coincide. Under these conditions

there exists neither a resultant force nor a resultant couple. The given forces are in equilibrium.

EXERCISE 67. Again change the force *de*, Fig. 30, in magnitude and position so that the given forces shall be in equilibrium. What becomes of the couple present in Ex. 66?

EXERCISE 68. Assume a horizontal force, a vertical force, and an oblique force and arrange them in three separate diagrams so as to illustrate the above three cases.

**Analytically.** — The given forces may be referred to any two rectangular axes. Let the sum of the $x$ and $y$ components of the given forces be $\Sigma X$ and $\Sigma Y$.

Then the resultant $R$ will have the magnitude

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2},$$

and its direction with the $X$-axis will be

$$\tan^{-1} \frac{\Sigma Y}{\Sigma X}.$$

If the sum of the moments of the given force with respect to the origin of coördinates is $\Sigma M$, then the intercept of the line of action of the resultant on the $X$-axis is

$$\frac{\Sigma M}{\Sigma Y}.$$

It follows that the given forces are in equilibrium when $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma M = 0$.

## SECTION VIII

## EQUILIBRIUM OF COPLANAR FORCES

### Graphical Solution

### I. EQUILIBRIUM OF THREE FORCES

Whenever three forces are in equilibrium their lines of action either intersect in a common point or are parallel.

EXERCISE 69. Demonstrate the above fact.

Under these conditions only two unknown quantities can exist in any definite problem. Either the magnitudes of two forces must be unknown or the magnitude and direction of one force must be unknown, all other quantities being given. Fig. 31 (a) and (b) illustrates a solution by means of the triangle of forces for non-

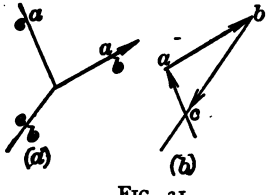


FIG. 31

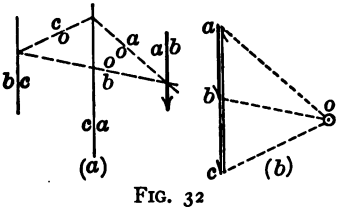


FIG. 32

parallel forces and Fig. 32 (a) and (b) illustrates a solution in the case of parallel forces by means of a funicular polygon.

In Fig. 31 (a) the known force *ab* and the directions of the forces *bc* and *ca* are given. The unknown magnitudes are determined in Fig. 31 (b) by the intersection of the lines *bc* and *ac* as shown.

In Fig. 32 (a) the magnitudes of two forces (*bc* and *ca*) are unknown. The funicular polygon, constructed by

means of the polygon of forces, in Fig. 32 (a) determines the direction of *oc* (the last line to be drawn); this in turn locates *c* in Fig. 32 (b).

Note that here again

*oa* intersects *ob* on *ab*,
*ob* intersects *oc* on *bc*,
*oc* intersects *oa* on *ac*,

and that both the polygon of forces and the funicular polygon "close" without the introduction of new forces.

EXERCISE 70. Assume two known forces and find a third wholly unknown force necessary for equilibrium,

(a) when the known forces are parallel,

(b) when the known forces are not parallel.

## II. EQUILIBRIUM OF FOUR FORCES

Whenever four forces are in equilibrium the resultant of any two of them must be equal in magnitude and opposite in direction to, and have the same line of action as, the resultant of the other two. This fact suggests the simple graphical construction for finding the unknown quantities in problems involving four forces in equilibrium illustrated in Fig. 33 (a) and (b).

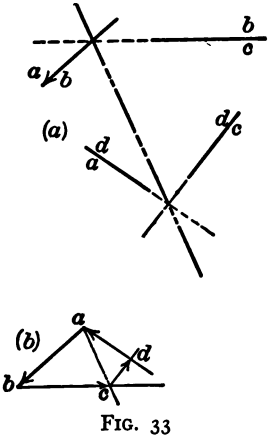


FIG. 33

Here one force, *ab*, is wholly known and only the lines of action of three other forces are given; thus the magnitudes of these three forces must be found.

The resultant of *ab*, the known force, and *bc* must pass thru the intersection of their lines of action. Also, the resultant of *cd* and *da* must pass thru the intersection of their lines of action. Therefore, the dotted-dashed line is the line of action of this resultant.

In Fig. 33 (b) starting with *ab* we can construct the triangle of forces *abc*; here *ac* (note the direction) is the resultant of *ab* and *bc*. Now *ca* (the reverse of *ac*) must be the resultant of *cd* and *da*, obtained from the second triangle of forces constructed upon *ca* as base. The magnitudes of *bc*, *cd*, and *da* are thus determined. As the four forces considered are in equilibrium the arrows representing them in Fig. 33 (b) follow each other about the polygon without reversal.

In a problem involving the equilibrium of four forces only three unknowns are permissible, if the problem is to be statically determinate.

EXERCISE 71. What other combinations of unknowns are permissible in the case of four forces in equilibrium? (Sketch and solve.)

### III. ANY SET OF FORCES IN EQUILIBRIUM

In the general case of a determinate problem in coplanar forces in equilibrium three unknown quantities are permissible.

Any force may have three unknown constituents, i.e.:

(a) its magnitude,

(b) the direction of its line of action,

(c) any point on its line of action.

Hence the following three important cases, involving three unknown quantities in problems concerning coplanar forces in equilibrium, may arise.

1. One force wholly unknown, all other forces fully determined.

2. One force unknown in magnitude, another force unknown in both magnitude and direction, but having given one point on its line of action, and all other forces fully determined.

3. Three forces all of unknown magnitude, but having their lines of action fully determined, and all other forces wholly determined.

The method of procedure in the graphical solution of any of the above problems is as follows:

(a) Letter the forces in the usual manner, i.e., *ab*, *bc*, etc., always ending the last force with *a*, thus *xa*, and always lettering the known forces first and the most unknown last.

(b) Construct as much of the polygon of forces as the given forces permit.

(c) Draw the possible rays from **any pole,** *o*, and construct the funicular polygon, *oa*, *ob*, etc.

(d) Draw the ray corresponding to *ox* (the last side of the funicular polygon) and by means of it close the polygon of forces, thus determining all unknown quantities.

In **connection with (c) the following directions must be carefully followed.** Thus in

**Case I,** in which nothing is known about the unknown force, the funicular polygon may be started anywhere and the intersection of the sides *oa* and *ox* determines a point on the unknown line of action.

**Case II,** in which one point on the line of action of the otherwise unknown force is known, the funicular polygon **must be started at this known point** and its final side *ox* must end at this same point. The ray parallel

to *ox* so determined will assist in closing the polygon of forces.

**Case III. The funicular polygon must be started at the intersection of the lines of action of the last-named unknown forces** and the next to the last side *ow* must end at this same point. The side *ox* will then be found to reduce to a point. The ray parallel to *ow* assists in the closing of the polygon of forces.

EXERCISE 72. Assume **two** wholly known forces and sufficient other forces to establish the three above cases and sketch the solution of each.

### Analytical Solution

In the analytical solution of any statical problem the magnitudes of the unknown forces should be represented by letters. A **force unknown in direction and magnitude** should always be replaced by a **horizontal and a vertical component.** The two unknown quantities then become magnitudes and the idea of direction, as such, is eliminated.

The solution of the equations

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma M = 0$$

will then yield the three unknown magnitudes.

In case only one force is wholly unknown the unknowns in the above equations will be two magnitudes (the components of the unknown force) and a distance which will serve to locate the position of this force relatively to the known forces.

# CHAPTER III

# TWO-DIMENSIONAL STRUCTURES

## SECTION IX

### ABUTMENTS AND THE FORCES THEY TRANSMIT

IN order to investigate the forces transmitted by the members of any structure the reactions of the abutments must usually be found.

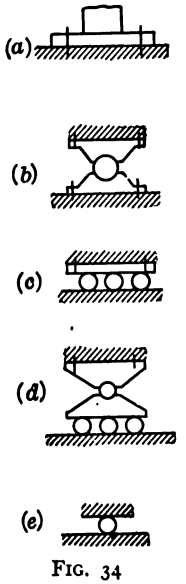

FIG. 34

These reactions are then considered as external, applied forces and form part of the total forces which the structure must resist.

Various forms of abutments are shown in Fig. 34. Sketch (a) illustrates a **Solid Anchorage**; (b) a **Pin or Hinge Abutment**; (c) a **Roller Abutment**; and (d) a **Pin-Roller Abutment** which may be replaced by (e), an abutment in which a single roller transmits the force. In the following discussion all frictional forces are neglected.

In an abutment such as illustrated in Fig. 34 (a), the force representing the reaction of the foundations upon the structure is wholly unknown. It may have any direction, its line of action may pass thru any point, and it may have any magnitude;

all this of course consistent with the strength of the parts involved.

The reaction of an abutment such as illustrated in (b) has one point on its line of action determined: it must pass thru the center of the pin-bearing. Otherwise, as regards magnitude and direction, it is undetermined. The reaction exerted thru a bearing such as (c) must be vertical (normal to the plates bearing the rollers) but it is otherwise unknown.

In (d) the reaction must not only be normal to the plates with which the rollers are in contact but must also pass thru the center of the pin bearing. Thus the only unknown quantity involved in the determination of this reaction is its magnitude.

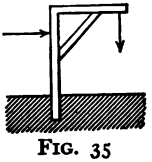

FIG. 35

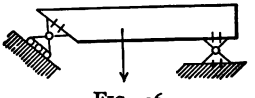

FIG. 36

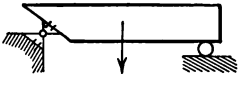

FIG. 37

EXERCISE 73. Show the bodies sketched in Figs. 35, 36, and 37 as free bodies and find graphically all external forces which must be supplied by the abutments for equilibrium.

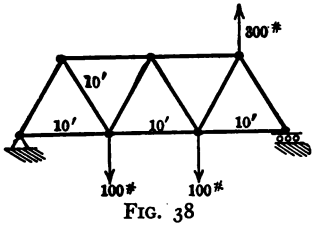


FIG. 38

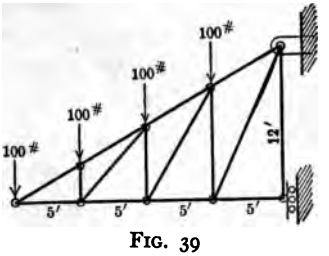


FIG. 39

EXERCISE 74. Find graphically and analytically the abutment reactions upon each of the frames loaded as shown in Figs. 38 and 39.

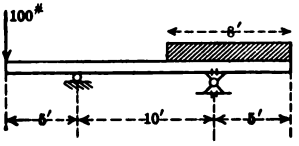


FIG. 40

EXERCISE 75. If the beam in Fig. 40 weighs 20 pounds per foot run and the uniformly distributed load is 10 pounds per foot run, find the reactions graphically and analytically.

## SECTION X

### IDEAL FRAMES

**A jointed frame** is a structure composed of members joined at **their ends only** by means of pins which allow hinge-like motions between connected members. To prevent this possible motion between the members the fundamental unit of which the frame is composed must be the triangle.

Frames satisfying the conditions just described are called **complete frames.** If the number of members is insufficient to prevent deformation (other than elastic deformation) the frame is called a **deficient frame.** In this case the frame will stand only if the joints are sufficiently stiff to resist any possible deformation. A frame having more members than are necessary to resist deformation is called a **redundant frame.** In such frames the lengthening or shortening of the redundant member (i.e., by means of a turnbuckle) will stress other members irrespective of any applied forces. The stresses in such frames depend greatly upon the accuracy in fitting the members. The computation of the stresses involves the elasticity and stiffness of the various members, so that they cannot be calculated by the principles of statics alone. Redundant frames are therefore classed with statically indeterminate structures.

**Ideal Frames.** — An ideal frame is a jointed frame whose pins are considered frictionless and which is loaded at the joints or pins only.

Thus the members of an ideal frame should properly be considered weightless. If their weight is to be taken into account it must be divided into two components and these must be added to the loads already acting on the pins at the ends of the member considered.

Any load to be carried at any point other than a pin should be borne by a beam which in turn distributes the load to the pins upon which it rests. Even if the load is not so carried it may for purposes of calculation be considered distributed as above described.

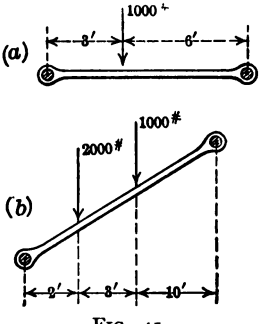

FIG. 41

EXERCISE 76. By what *equivalent pin loads* can the loads shown in Fig. 41 (a) and (b) be replaced?

**The members of ideal frames are subject to tension and compression only; when in tension they are called ties, and when in compression, struts.**

The stresses in the members of ideal frames may conveniently be found by the following methods:

**I. The Method of Sections.**

(a) Ritter's Method (analytical).

(b) Culmann's Method (graphical).

**II. By Means of Stress Diagrams,** also known as Maxwell or Cremona diagrams.

## I. Method of Sections

This method is particularly valuable when the stres in only one given member of the frame is required. I derives its name from the fact that the frame is con ceived as divided into two parts by an imaginary cut o "section." Either portion may then be treated as a free body in equilibrium. The "**section**" dividing the frame into two parts **should never cut more than three members whose stresses are unknown,** for otherwise more than the allowable number of unknowns would be introduced into the problem.

### (a) Ritter's Method

The analytical solution can best be illustrated by a concrete problem.

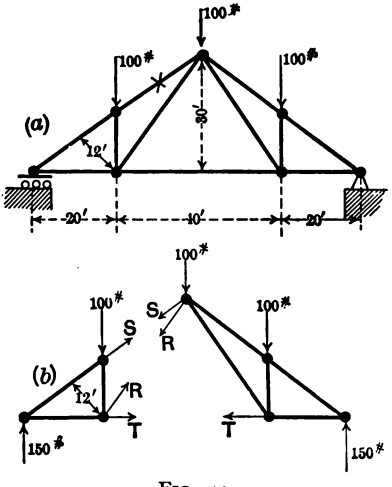


Fig. 42

Let Fig. 42 (a) represent the frame and its loading and let it be required to find the stress in the marked (×) member.

A section should be passed cutting this marked member and separating the whole frame into two parts, but this section must not cut more than two other members, three in all.

The resulting parts of the frame are shown as **free bodies** in Fig. 42 (b). Here besides the **loads and the**

**reactions are shown the forces** ($S$, $R$, $T$) **replacing the cut members.** The reactions of the abutments must be found by calculations made upon the frame as a whole, considered as a free body.

As both parts of the divided frame are in equilibrium under the action of the forces shown, either part may be used to calculate the magnitude of $S$, the required stress. For convenience use the simpler left-hand part.

**Taking moments about the intersection of the lines of action of the forces R and T** (Why?),

we have $\Sigma \text{ moments} = -12S - 3000 = 0,$

where the arm (12 feet) of the force $S$ may either be computed or it may be scaled from an accurately drawn diagram of the frame.

Thus, $S = -250$ pounds.

**To interpret the minus sign in S = − 250 pounds,** note that the arrows representing the forces transmitted by the cut- members (Fig. 42 (b)) **indicate tensions** in all of the cut- members, for these forces indicate **pulls upon the pins involved.** They should always be so drawn. The minus sign shows that in the case of the force $S$ the direction of this arrow was incorrectly assumed; in fact it should have been reversed. Therefore the **forces S actually push upon the pins involved** and the member replaced by the forces $S$ is in **compression** to the extent of 250 pounds.

A **minus** sign always **indicates compression** and a **plus** sign always **indicates tension,** provided the arrows are always originally assumed as pulling upon the pins involved.

EXERCISE 77. Show that the other rods cut in Fig. 42 (b) are both in tension, one to the extent of 121 pounds and the other to 133 pounds.

EXERCISE 78. Find the stresses in the other horizontal members of the frame shown in Fig. 42.

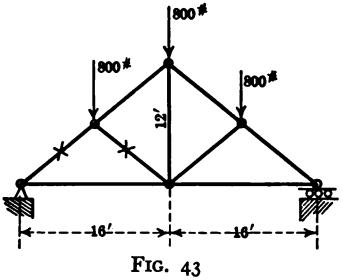

FIG. 43

EXERCISE 79. What are the stresses in the vertical members of this same frame?

EXERCISE 80. Find the stresses in the marked (X) members of the frame shown in Fig. 43.

EXERCISE 81. Find the stress in the central lower horizontal member of the frame shown in Fig. 38.

EXERCISE 82. What are the stresses in the vertical and horizontal members nearest the wall in Fig. 39?

## (b) Culmann's Method

Culmann's Method is a graphical method for obtaining the stress in any member of an ideal frame by means of the method of sections.

To illustrate consider the stress in the marked member in Fig. 42 (a).

Fig. 44 (a) shows the frame as a free body and by means of the force polygon, Fig. 44 (b), and its funicular polygon the reactions *de* and *ea* have been determined. Now cut the frame into two parts, supplying the forces $S$, $R$, and $T$ to replace the cut members and consider the equilibrium of the left-hand part of the frame.

The external applied forces *ab* and *ea* can be replaced by their resultant *eb* whose line of action passes thru the

intersection of the sides *ob* and *oe* of the funicular polygon. In this manner the forces acting upon that part of the frame considered may always be reduced to four in number; in this case to the resultant, *eb*, and *S*, *R*, and *T*.

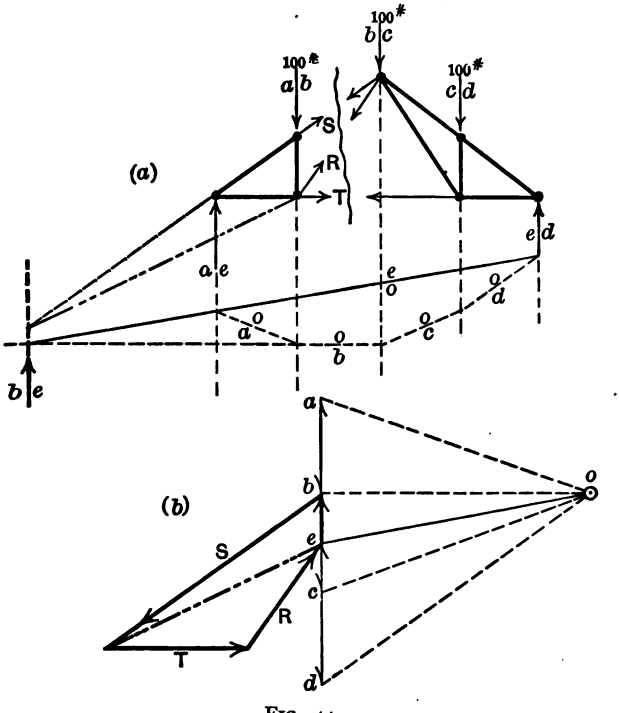


FIG. 44

The solution can now be completed, by the method outlined under the equilibrium for four forces, as indicated by the heavy lines in the figure. These results may be obtained just as readily by considering the right-hand part of the frame as a free body. Do this.

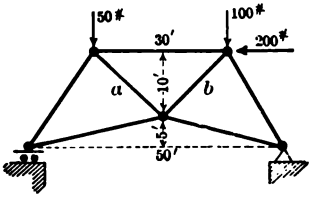


FIG. 45

EXERCISE 83. Find the stresses in members (a) and (b) in Fig. 45, by Culmann's method.

## II. STRESS DIAGRAMS

The method for finding the stresses in the members of ideal frames by means of stress diagrams, also called Maxwell or Cremona diagrams, is particularly useful when the stresses in all the members of the frames are desired. It should also be noted that this method serves as a check upon itself so that serious errors are not likely to pass unobserved.

As in all graphical work, notation is here of prime importance. In Fig. 46 (a) the notation already used in connection with funicular polygons is applied to the members and to the external or applied forces of a frame.

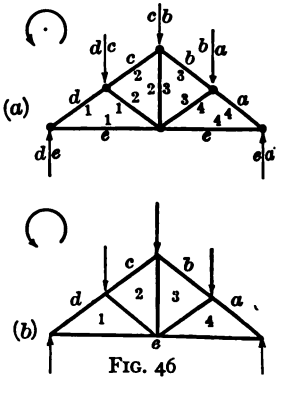


FIG. 46

Thus the applied forces are: the loads, *ab*, *bc*, *cd*, and the reactions of the abutments, *de* and *ea*. Here the forces are all lettered counterclockwise about the frame. Applying the same notation to the members of the frame we might designate the rafters as 1*d*, 2*c*, 3*b*, and 4*a*; the horizontal members would be 1*e* and 4*e*; the remaining members are 12, 23, and 34. Note that as few numbers and letters as possible have been used and to avoid repetition of numbers and letters the notation shown in Fig. 46 (b) will be adopted.

In the above notation any pin is named by stating in succession the names of all areas surrounding it and always in the same rotation as is used in naming the applied forces. Thus the pin at the apex of the frame in Fig. 46 (b) is *23bc2* or *bc23b* but not *2cb32*, for a counter-clockwise rotation was originally used in naming the applied forces.

**Not only does bc23b** signify a pin of the above frame **but it also designates all forces acting upon this pin.** Thus the forces acting upon the apex pin are *bc*, *c2*, *23*, *3b*, the first being an external force and the last three being the forces transmitted to this pin by the members joined by it.

Again, the forces acting upon the pin over the left abutment are *de*, *e1*, and *1d*; or, more simply, *de1d*.

The naming of the forces acting upon any pin should always begin with the known forces and end with the unknown forces of which there should never be more than two. This is essential, for the forces acting upon any pin will always be concurrent forces in equilibrium. As their polygon of forces must close the known forces must be laid down first, and not more than two unknown forces can be determined as regards magnitude by the closing of this polygon.

Consider now the equilibrium of the various pins in the frame shown in Fig. 47. Here each pin is shown as a free body, the forces acting upon it are named, and its polygon of forces is shown. Finally the corresponding stress diagram is shown. This simply consists of the separate polygons of forces in adjacent positions, thus avoiding unnecessary repetition of lines. The **separate polygons of forces are never drawn** in applying this

method of solution; the stress diagram should be constructed directly from the diagram of the frame with the help of the notation explained above. The dotted lines in the stress diagram of Fig. 47 indicate the portion not shown in the separate polygons of forces above it.

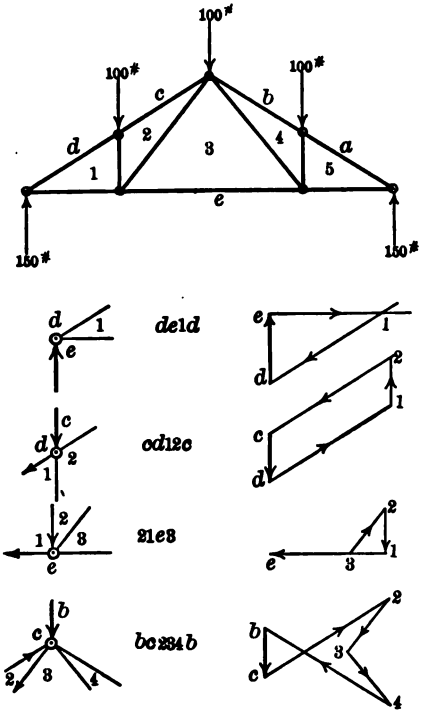


FIG. 47

In stress diagrams the arrow-heads upon the lines showing the stresses in the members should be omitted, for in the complete figure each of these lines would have two arrow-heads in opposite directions.

To discover from the stress diagram whether any member is in tension or compression proceed as follows:

Consider the member *c2* or *2c*. If its action upon pin *bc234b* is considered its name is *c2* (and not *2c*). Referring now to the stress diagram, the

direction c2 shows that this member is **pushing upon pin bc234b**; thus this member must be in compression. The same member considered as acting upon pin **cd12c** is now known as *2c* (not *c2*); again referring to the stress diagram the direction *2c* shows that this member is **pushing upon pin cd12c**; thus again indicating compression in the member.

As another illustration consider member *3e*. Its name in connection with pin *1e321* is *e3*. In the stress diagram *e3* indicates a force acting towards the right; thus member *e3* **pulls** the pin *1e321* **towards the right** or the member must be **in tension.**

EXERCISE 84. Sketch the stress diagram for the frame shown in Fig. 46 assuming loads of 200 pounds as shown and indicate ties with plus signs and struts with minus signs.

EXERCISE 85. Find graphically the reaction of the abutments for the frame shown in Fig. 48. Then re-letter the

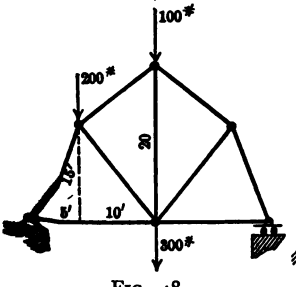


FIG. 48

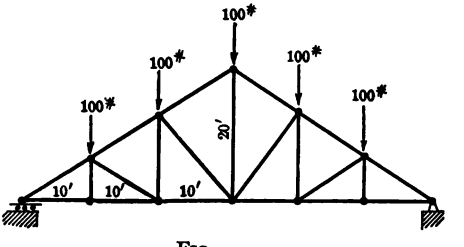


FIG. 49

frame diagram and draw its stress diagram. Indicate tension and compression in the members with + and − signs respectively.

EXERCISE 86. Check the results of Ex. 80 by means of a stress diagram.

EXERCISE 87. Check Ex. 81 by means of a stress diagram.

EXERCISE 88. Find the stresses in all members of the frame illustrated in Fig. 49.

The French or Fink Truss illustrated in Fig. 50, much used as a roof truss, offers some difficulty in the construction of its stress diagram.

If the diagram is started with the forces acting on the pin at the left abutment then after three pins have been considered further progress is impossible by a continuation of the same process, because three unknown forces occur at each of the pins next in order. To continue

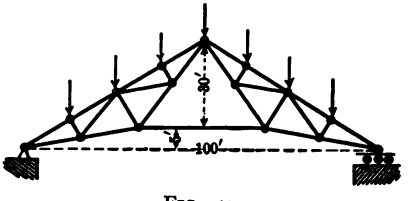


FIG. 50

the stress diagram, the stress in some member may be found by some independent method and it may then be regarded as known. Thus the stresses in the horizontal tie rod may be found by the method of sections and then the pin upon which it acts will have only two unknown forces acting upon it.

EXERCISE 89. Draw the stress diagram for the Fink truss of Fig. 50 assuming the loads to be 100 pounds each.

The stresses due to oblique loading (due to wind) are found in exactly the same manner as above described. It will be only necessary to call attention to a simple method of finding the abutment reactions graphically. The oblique loading due to wind is considered normal to the surface of the roof and of course occurs only on one

side of it at any time. In Fig. [illegible] the wind load on any one panel of the roof is [illegible]. This pressure is distributed to the [illegible] of the rafters. As shown in the figure the end [illegible] the load on one half of a panel.

The abutment at the right can furnish a vertical reaction only; the one at the left a reaction in any direction. The lettering of the forces should then be started as indicated and run clockwise about the frame.

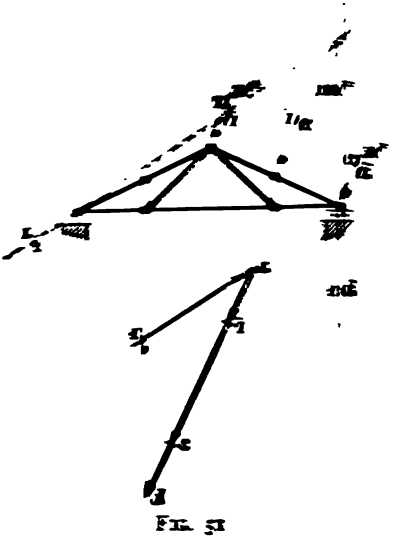

Fig. 51

Now instead of applying the funicular polygon to find these reactions, the resultant of the loads can readily be found and it together with the two reactions constitutes a system of three forces in equilibrium. Thus the intersection of [illegible] (the line of action of the resultant load) and *de* serves to locate the direction of the left end reaction, and a triangle of forces can then be constructed to determine the magnitudes of the reactions.

It should be carefully noted that the resultant load can only be used to determine the reactions and not to find the stresses.

Exercise 90. Draw the stress diagram for the frame loaded as shown in Fig. 51.

EXERCISE 91. Assume the wind pressure to act upon the left slope of the roof in Fig. 51 and find the stresses induced in the members of the frame.

In roofs of short span it is usual to anchor both ends of the truss, making no special allowance for expansion. Under these conditions the reactions at the abutment are indeterminate as four unknown quantities now occur: a magnitude and a direction at each end. In such cases the reactions at both ends are **assumed** parallel to the wind-load forces. They can then be determined by means of a funicular polygon and the stress diagram can be constructed in the usual manner.

EXERCISE 92. Assume both ends of the truss in Fig. 51 fixed firmly to the supporting wall and construct the stress diagram.

EXERCISE 93. Assume a wind pressure of 100 pounds per panel on the right slope of the frame in Fig. 49 and sketch the stress diagram due to the wind pressure only.

EXERCISE 94. (a) Two ideal frames are similar in all respects but one is twice as large as the other. If their pin loads are the same, how do the stresses in the members compare?

(b) Two ideal frames are identical in all respects but the pin loads on one are twice the loads on the other. How do the stresses in the various members of one compare with the stresses in the members of the other?

## SECTION XI

### FRAMES IN GENERAL

In this section will be considered the forces acting upon the members of frames not necessarily loaded at the pins and in which the members may be continuous thru the pins connecting the various members.

Such frames are used in the construction of wharf and foundry cranes, derricks, etc.

In the discussion of frames of this kind it should be remembered that members acted upon by forces at other points than the ends may be subject to bending stresses as well as simple tensions or compressions. Such members do not transmit forces along their center lines.

As an example consider the crane illustrated in Fig. 52. This crane is supported by a foot-step bearing at *A* and a collar bearing at *B*, and supports a load of 8000 pounds, slung from the pin *C* without pulleys or tackle. Assume the weight of the jib to be 1500 pounds and the weight of the post 1200 pounds. The forces acting upon each of the members of this frame are to be found. A knowledge of these forces is necessary in designing the various members of the frame.

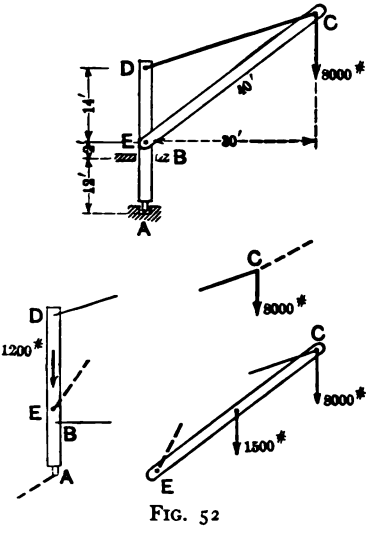

FIG. 52

In a problem such as this the members cannot be assumed to be in simple tension or compression, as was the case in the ideal jointed frame. The jib *EC*, for instance, is subject to forces (its weight) not acting at its ends, therefore the forces with which the jib acts upon the pins at its ends will not have the center line of the jib as line of action. In this frame only the tie-rod *CD*

is in simple tension. It alone acts upon the pins at its ends in the direction of its length. Even this last statement is true only under the assumption that the weight of the tie-rod is neglected; this may be done because its weight is small in comparison to the force it transmits.

To find the forces acting on the various members of this crane make free bodies of them as shown in Fig. 52. The pin $C$ considered as a free body has acting upon it the load 8000 pounds, a force in the direction of the tie-rod unknown in magnitude, and a force transmitted thru the jib unknown in both direction and magnitude as indicated by the dotted line. The forces at $C$ cannot be determined from it alone as there are too many unknown quantities. Consider next the post $DA$ as a free body. On it, at $D$ and $B$, act two forces unknown in magnitude and, at $A$ and $E$, two forces unknown in both magnitude and direction, besides its known weight of 1200 pounds. Again we have too many unknowns to permit solution.

The jib, however, has acting upon it only forces involving three unknown elements, i.e., a force of unknown magnitude at $C$ (the tension in the tie-rod) and a force unknown in direction and magnitude at $E$ (the force transmitted to the jib by the post). Applying the graphical constructions for equilibrium to the forces acting on the jib the unknowns can be determined.

Now return to the post as a free body and note that the forces at $E$ and $D$ are equal in magnitude and opposite in direction to the forces just found; then the forces acting on the post at $B$ and $A$ may be found by another graphical construction.

EXERCISE 95. Solve the above problem as indicated.

This problem may also be solved graphically by considering the whole structure as a free body and so finding the forces acting upon the post at $A$ and $B$. Then consider the jib as a free body and thus find all other required forces.

EXERCISE 96. Find the reactions of the abutments for the crane shown in Fig. 52.

EXERCISE 97. (a) Find these reactions if the members of the crane (Fig. 52) are considered weightless.

(b) Find graphically the tension in the tie-rod and the compression in the jib under the conditions of (a).

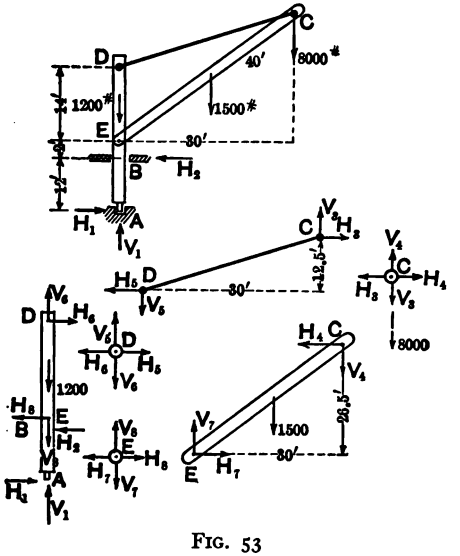


FIG. 53

The following analytical solution of the crane already discussed illustrates a general method of solution which may be applied to any frame.

In Fig. 53 the whole frame is shown as a free body.

From it the reactions $H_1$, $V_1$, and $H_2$ of the abutments at $A$ and $B$ can be found.

By moments about $A$

$$12\,H_2 = 240{,}000 + 22{,}500,$$

or $$H_2 = 21{,}900 \text{ pounds.}$$

The sum of the horizontal forces gives

$$H_1 - H_2 = 0 \quad \therefore \quad H_1 = 21{,}900 \text{ pounds.}$$

The sum of the vertical forces gives

$$V_1 - 1200 - 1500 - 8000 = 0,$$

or $$V_1 = 10{,}700 \text{ pounds.}$$

In Fig. 53 is also shown each part of the frame as a free body. The pin $C$ has acting upon it the load 8000 pounds, the forces $H_3$ and $V_3$ due to the tie-rod $DC$, and $H_4$ and $V_4$ due to the jib $EC$. At $C$ on the tie-rod act $H_3$ and $V_3$, equal and opposite forces to $H_3$ and $V_3$ on the pin $C$; these represent the action of the pin $C$ upon the tie-rod, etc.

As each part of the frame is in equilibrium the following equations result from

$$C\begin{cases} H_3 = H_4 & (1) \\ V_3 + 8000 = V_4 & (2) \end{cases}$$

$$D\begin{cases} H_5 = H_6 & (3) \\ V_5 = V_6 & (4) \end{cases}$$

$$E\begin{cases} H_7 = H_8 & (5) \\ V_7 = V_8 & (6) \end{cases}$$

$$DC\begin{cases} H_5 = H_3 & (7) \\ V_5 = V_3 & (8) \\ V_3(30) = H_3(12.5) & (9) \end{cases}$$

$$EC\begin{cases} H_4 = H_7 & (10) \\ V_7 = 1500 + V_4 & (11) \\ V_4(30) + 1500(15) = H_4(26.5) & (12) \end{cases}$$

s the problem still involves 12 unknown forces and e are 12 equations involving these unknown forces, problem may be solved without the additional three ations obtainable from the post $DA$. This is due to fact that three unknown forces $H_1$, $V_1$, and $H_2$ have ady been obtained by considering the whole frame as ee body.

he equilibrium of the forces acting upon the post $DA$ serve as a check upon the solution of the above 12 ations.

eferring to the above equations, by combining equa- s (1), (2), (9), and (12) the following equation in- ving only $V_4$ can be obtained,

$$30\,V_4 + 22{,}500 = \left\{\frac{30\,(V_4 - 8000)}{12.5}\right\} \quad (26.5),$$

nce $V_4 = 15{,}810$ pounds.

m (2) $V_3 = 7{,}810$ pounds.

m (11) $V_7 = 17{,}810$ pounds.

ıs, $V_3 = V_5 = V_6 = 7810$ pounds,

$$= H_4 = H_5 = H_6 = H_7 = H_8 = \frac{V_3\,(30)}{12.5} = 18{,}750 \text{ pounds},$$

$V_4 = 15{,}810$ pounds,

$V_7 = V_8 = 17{,}810$ pounds.

XERCISE 98. Show that the post in Fig. 53 is in equi- um under the action of the forces found above.

XERCISE 99. How can the tension in the tie-rod be ined from a single summation of moments?

XERCISE 100. If a counter-weight of 20,000 pounds, se center of gravity is 9 feet to the left of the post in Fig. rests upon a horizontal member 12 feet long, which is ened to the pin $E$ and has its left end supported by a tie-

rod fastened to the pin $D$, find the forces then actin the post.

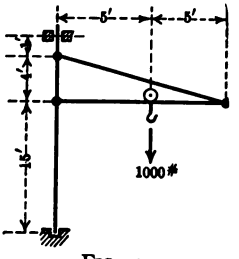

FIG. 54

Compare the forces acting u post with and without counter and state the advantages of the c weight.

EXERCISE 101. Consider th bers of the crane, Fig. 54, to be less and find the forces acting up member (a) graphically, (b) analy for the position of the load shown.

Fig. 55 (b) shows a crane supported by a foot st a crown of rollers. As shown, the post is enlarged form of a frustum of a cone and rollers or wheels are set with their axes coinciding with the element of the conical surface.

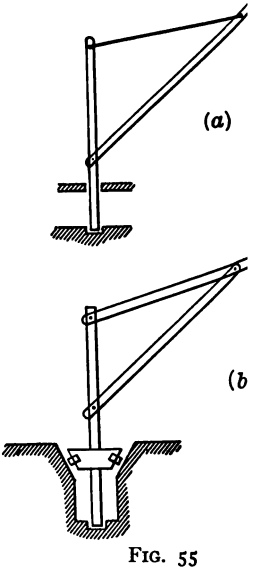

FIG. 55

EXERCISE 102. Sketch a graphical solution showing (a) the reactions, (b) all forces acting upon all members of the cranes shown in Fig. 55. Assume all members weightless. What effect has the crown of rollers upon the foot-step reactions?

In Fig. 56 is shown a **crane with chain.** If the load $W$ pounds is supported by a single movable pulley then the tension thruout the chain is $\frac{W}{2}$, all friction negl

:never any part of the crane is represented as a
the action of the chain upon it must be replaced
e of $\frac{W}{2}$ pounds whose line of action coincides
direction of the chain at its point of contact
part considered.

upper end of the jib the action of the chain
represented by three forces of $\frac{W}{2}$ pounds, two
are vertical, one passing thru the pin and one
o the pulley, and the
ng along the dotted
rds the pulley, $A$, on

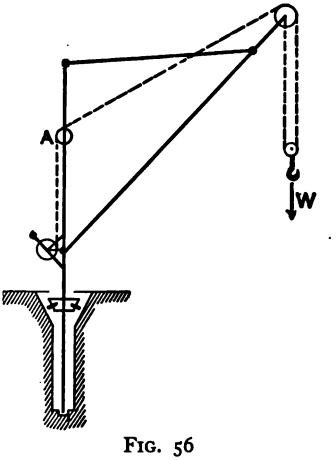


FIG. 56

following exercises
)ers of the crane are
ımed weightless and
ısions of the pulleys
neglected, i.e., all
ting tangent to the
ıay be assumed to
their centers. This
/ done in practice
: diameters of the
s compared to the dimensions of the members,

SE 103. Find graphically the reactions of the
; of the crane shown in Fig. 56.

SE 104. Find all forces acting upon the jib of

SE 105. Show the post of this crane as a free
from it find all forces acting upon the post.

EXERCISE 106. What effect has the position of the pulley *A* (Fig. 56) upon the forces acting upon the various members of the crane?

Large cranes for heavy work are usually designed so that they may be treated as ideal frames, in so far as the weight of the members is neglected. In Fig. 57, the crane is supported on a circular base which may rotate about a central pin and which rests upon rollers running on a circular track having this pin as center. The boilers, engines, and counter-weight may be considered as loads of 100 tons and 80 tons as shown. The central pin is anchored to the foundations by means of heavy bolts. The load of 150 tons is supported by a wire rope passing over four sheaves in the lower block and four sheaves in the upper block, thence, as shown, to the hoisting drum.

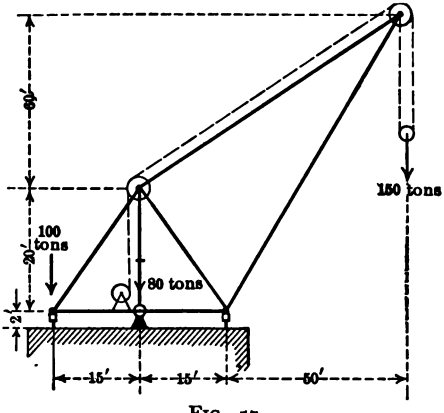

FIG. 57

EXERCISE 107. Find the forces exerted on the foundations by the crane shown in Fig. 57, (a) when supporting a load of 150 tons, (b) without load.

EXERCISE 108. Sketch the stress diagram for this crane assuming the members weightless and all forces acting at the pins. The load supported is 150 tons.

# CHAPTER IV

# SOLID STATICS

## SECTION XII

### THEORY AND APPLICATION

IN this section the equilibrium of forces acting in three-dimensional space will be considered.

We shall follow the method given for coplanar forces in Vol. I of this Text-Book of Mechanics and apply it to forces acting in three dimensions.

In Fig. 58 consider any force $F$ whose line of action passes thru the point $(x, y, z)$. It can be resolved into three components $X$, $Y$, and $Z$ parallel respectively to the three axes of reference.

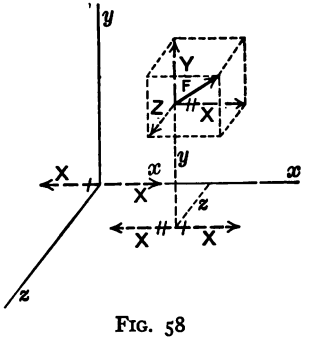


FIG. 58

Without disturbing the equilibrium of the system, two forces equal in magnitude and having the same line of action but being opposite in direction can be applied at the point $(x, o, z)$. Assume these forces equal to $X$ and parallel to the $x$-axis. Apply a similar pair of forces at the origin as shown.

Thus, instead of the component $X$ of the force $F$, five forces, equivalent to $X$, may be used. Four of these form two couples, leaving one force acting at the origin

in a positive direction along the $x$-axis. One of these couples (whose forces are marked /) lies in the plane $xz$, the other couple (whose forces are marked //) lies in a plane parallel to the plane $xy$.

The signs of these couples are determined by the direction of rotation which they tend to impart to the coördinate axis which is perpendicular to their plane of action. If this rotation is such as would advance a right-handed screw in the positive direction along the coördinate axis then the couple is positive.

Thus the moments of the above couples are

$$-yX \text{ about the } z\text{-axis},$$
$$+zX \text{ about the } y\text{-axis}.$$

Treating the $Y$ component of $F$ in a similar manner its action can be reduced to a positive force along the $y$-axis and to two couples whose moments are

$$+xY \text{ about the } z\text{-axis},$$
$$-zY \text{ about the } x\text{-axis}.$$

(Draw a diagram, similar to Fig. 58, illustrating these couples.)

By treating the $z$ component of $F$ in a similar manner its action may be reduced to a positive force along the $z$-axis and to two couples whose moments are

$$yZ \text{ about the } x\text{-axis},$$
and $$-xZ \text{ about the } y\text{-axis}.$$

(Draw a diagram, similar to Fig. 58, illustrating these couples.)

As any force may be treated in the same manner as above illustrated for the force $F$ it follows that any number of forces may be reduced to the following three forces:

$\Sigma X$ along any arbitrarily chosen $x$-axis,
$\Sigma Y$ along any arbitrarily chosen $y$-axis,
and $\Sigma Z$ along any arbitrarily chosen $z$-axis;

and to a set of three **couples** whose **moments** are

$\Sigma (yZ - zY)$ about the above $x$-axis,
$\Sigma (zX - xZ)$ about the above $y$-axis,
and $\Sigma (xY - yX)$ about the above $z$-axis.

For **equilibrium of translation** it follows that

$$\Sigma x = 0, \quad \Sigma y = 0, \quad \text{and} \quad \Sigma z = 0.$$

For **equilibrium of rotation** it follows that

$$\Sigma (yZ - zY) = 0, \ \Sigma (zX - xZ) = 0, \ \text{and} \ \Sigma (xY - yX) = 0.$$

Combining Fig. 58 with the similar figures sketched by yourself and referring to Varignon's Theorem of Moments (Vol. I of this text-book, page 40), it becomes evident that the "moments about the various axes," i.e., $\Sigma (yZ - zY)$, etc., may readily be obtained by taking the sum of the moments of the projections of the forces upon the coördinate planes about the projection on the same plane of the axis normal to it as origin of moments.

As an example illustrating the application of the above **analytical** principles of three-dimensional equilibrium consider the shaft represented in Fig. 59. A vertical force of 100 pounds acts upon a horizontal crank 10 feet long; the magnitude of a horizontal force acting on a vertical crank 5 feet long and the reactions of the bearings are required.

As no force or component force acts along the length of the shaft the only forces which the bearings can be called upon to exert upon the shaft will be the forces $X_1$, $Y_1$ at the front bearing and $X_2$, $Y_2$ at the rear bearing.

Let the directions of these reactions be assumed as shown in Fig. 59.

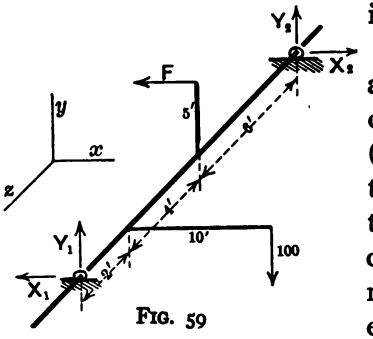


FIG. 59

As the system of forces acting upon the shaft and cranks as a free body (Fig. 59) is in equilibrium, the sum of the forces (or their components) in the direction of the three axes must each separately equal zero. Thus,

$$\Sigma X = 0 \text{ gives } X_2 = F + X_1 \quad . \quad . \quad . \quad (1)$$

$$\Sigma Y = 0 \text{ gives } Y_1 + Y_2 = 100 \quad . \quad . \quad . \quad (2)$$

and $$\Sigma Z = 0 \text{ gives } 0 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also the sum of the moments of the projections of all forces upon a plane parallel to the $yz$-plane about any point must be zero. Assuming the projection of the forward bearing as the origin of moments, then

$$\Sigma M_{yz} = 0 \text{ gives } 12\,Y_2 = 2\,(100). \quad . \quad . \quad (4)$$

Similarly the sum of the moments of the projections of all forces upon a plane parallel to the $xz$-plane must be zero. Assuming the projection of the forward bearing as the origin of moments,

$$\Sigma M_{xz} = 0 \text{ gives } 6\,F = 12\,X_2. \quad . \quad . \quad . \quad (5)$$

Finally, projecting all forces upon a plane parallel to the $xy$-plane and taking moments about the projection of the forward bearing,

$$\Sigma M_{xy} = 0 \text{ gives } 5\,F = 10\,(100). \quad . \quad . \quad (6)$$

Solving equations (1) to (6), we have

$F = 200$ pounds, $X_2 = 100$ pounds, $Y_2 = 16.6$ pounds, and $X_1 = -100$ pounds, $Y_1 = 83.4$ pounds.

Note that $X_1 = -$ 100 pounds, the minus sign indicating that the force $X_1$ as assumed in Fig. 59 should be reversed in direction, or the front bearing actually pushes the shaft towards the right with a force of 100 pounds.

The above solution shows that a problem in solid statics really involves three problems in plane statics, each dealing with the projections of the given forces upon one of three mutually perpendicular planes.

Thus, equations (1), (2), and (6) establish the equilibrium of the forces projected on the $xy$-plane, equations (2), (3), and (4) establish the equilibrium of the forces projected on the $yz$ plane, and equations (1), (3), and (5) establish the equilibrium of the forces projected on the $xz$-plane.

This idea is made use of in the graphical treatment of problems in solid statics, as is shown in the following **graphical solution** of the problem illustrated in Fig. 59.

In Fig. 60 the shaft and cranks together with **all forces** acting upon them **are represented in three projections.**

In the lettering used the projections of forces are designated as follows: In the **plan** by the plain letters, in the **elevation** by the same letters primed, and in the **profile** by the same letters double primed.

The known force of 100 pounds, called $AB$, is designated in the plan by $ab$, in the elevation by $a'b'$, and in the profile by $a''b''$; the unknown forces are $BC$, $CD$, and $DA$. The unknown forces contain five unknown elements, i.e., the magnitude of $BC$, the magnitude and direction of both $CD$ and $DA$. The forces $CD$ and $DA$ are determined as far as their planes of action are concerned; also one point upon the line of action of each is known.

Referring to the profile, Fig. 60, it will appear that in this projection only three forces need be considered:

$c''a''$, replacing the projections $c''d''$ and $d''a''$ of $CD$ and $DA$; thus, $b''c''$ can be found by a triangle of forces.

Passing to the plan, where $bc$ is now known and $ab$ is zero, $cd$ and $da$ can readily be found by means of a funicular polygon.

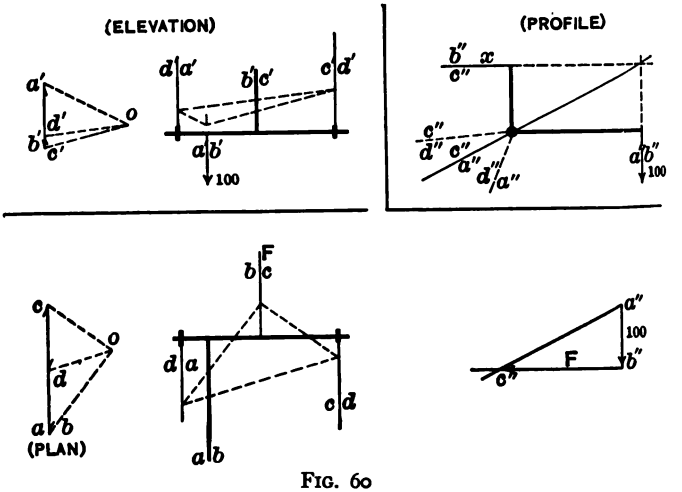


FIG. 60

Finally, in the elevation, where $a'b'$ is known and $b'c'$ is zero, we find $c'd'$ and $d'a'$.

Thus the horizontal and vertical components of the reactions at the bearings are determined.

EXERCISE 109. Fig. 61 represents a vertical shaft supported by bearings at $A$ and $B$, and by a foot step at $C$. $GFH$ is a helicoidal surface, such as the surface of a square-threaded screw, upon which rolls the wheel $F$, which is fastened to the rod $DE$ supported by bearings at $D$ and $E$ and supporting the load $W$.

Assuming the weight of the shaft $AB$ together with the cam and pulley to be 300 pounds, with the center of gravity in the axis $AB$, and the pitch angle of the helix as arc tan $\frac{1}{4}$,

ıd the load $W$ and the reactions at all the bearings when ıe system is in equilibrium (a) analytically, (b) graphically. Base the solution upon two free bodies.)

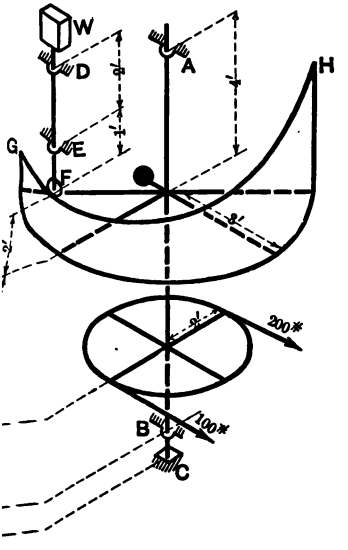


FIG. 61

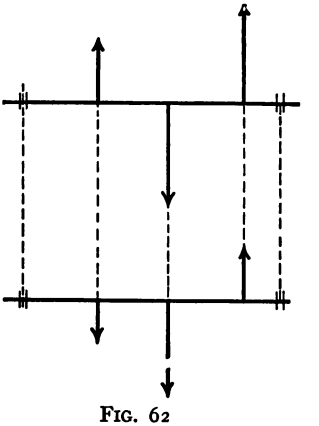

FIG. 62

EXERCISE 110. Fig. 62 shows the plan and elevation of shaft supported by two bearings. If the arrows represent he projections of three forces acting upon the shaft, find raphically the forces acting on the bearings.

Construct a profile showing all forces acting upon the shaft.

## SECTION XIII

### CONCURRENT FORCES IN SPACE—APPLICATIONS TO STRUCTURES

The case of concurrent forces in space is usually andled with greater simplicity by the special method ow to be described than by the general method which

would follow from an abridgment of the methods discussed in the previous section.

Whenever concurrent forces in space are in equilibrium, the sum of the component forces along any three axes must each separately be equal to zero, i.e.,

$$\Sigma X = 0; \quad \Sigma Y = 0; \quad \text{and} \quad \Sigma Z = 0.$$

Thus only three unknowns are possible and these are usually unknown magnitudes, the directions of the lines of action of the unknown forces being known.

A method for the determination of these unknown magnitudes is as follows:

(a) Find the resultant of all the known forces.

(b) Find the intersection of the plane determined by this resultant and the line of action of any one of the unknown forces with the plane determined by the lines of action of the two remaining unknown forces.

(c) Treat this intersection as the line of action of an imaginary unknown force and by the principles of plane statics find its magnitude and the magnitude of the unknown force first selected.

(d) Now, by plane statics, divide the force along the imaginary line of action into two components along the lines of action of the other two unknown forces originally assigned.

To illustrate, consider as an application the finding of the compressions in the legs of

**A Tripod Supporting a Weight.** — Fig. 63 (a) represents the tripod in perspective, and Fig. 63 (b) its development upon the horizontal plane drawn to scale. Let $a$, $b$, and c represent the lengths of the legs of the tripod and $m$, $n$, and $p$ the distances between their feet. By

ıs of the intersection of the perpendiculars dropped *MNP* upon the sides *m*, *n*, and *p* of the base of the d the lines *y* and *x* are determined. Referring to 53 (a) it will be noted that *x* is the **imaginary leg** of ripod which replaces legs *b* and *c* and lies in the plane and the line of action of *W*. Fig. 63 (c) shows the

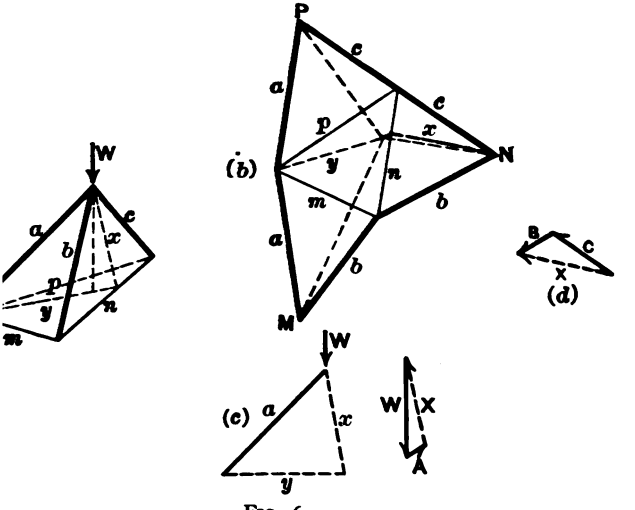


FIG. 63

gle formed by *a*, *x*, and *y*, in its true size; the cor- ıding triangle of forces determines the stresses *A* *X* in the legs *a* and *x* respectively.

ɔw from Fig. 63 (b), in connection with Fig. 63 (d) *B* ⁻ are found to be the components of *X* along the legs l *c* respectively.

more **general solution** of the above problem which .es equally well whether the load is vertical or not ɔased upon the method used above is furnished by ıpplication of descriptive geometry.

In Fig. 64, $a'$, $b'$, and $c'$ represent the elevation of the tripod legs and $a$, $b$, and $c$ their plan. $F'$ and $F$ are the projections of the applied force.

The imaginary leg replacing the legs $b$ and $c$ is represented by $x'$ and $x$, determined by the intersection of the planes of the lines $a$ and $F$, and $b$ and $c$.

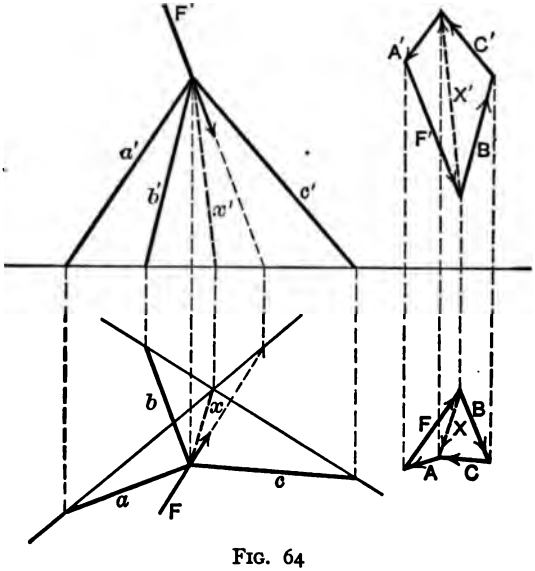


FIG. 64

To the right are shown first the projections of the triangle of forces for the equilibrium of the forces in the plane determined by the leg $a$ and the applied force $F$; then the resolution of the force $X$ into the forces $B$ and $C$, also in projection.

The quadrilaterals formed by the forces $A'$, $B'$, $C'$, and $F'$ and $A$, $B$, $C$, and $F$ are thus the projections of the space polygon representing the forces acting upon the pin at the apex of the tripod.

find the magnitudes of the forces transmitted
the legs of the tripod the true length of the lines
projections are $A$, $A'$; $B$, $B'$, etc., must be found.

ERCISE 111. Are all the legs of the tripod, loaded as
in Fig. 64, in compression? Give the reason for your
.

ERCISE 112. Assume the projections of four concur-
rces and find the projections of their resultant.

ERCISE 113. Find the stresses in the legs of a tripod
ting 1000 pounds, if each leg is 10 feet long and the
f the equilateral triangle forming its base are each 5
ıg. (Solve both analytically and graphically.)

ERCISE 114. A tripod has two of its legs braced to-
so as to carry a winch which is used in connection with
ınd tackle for lowering water mains. The legs of the
are each 10 feet long and their feet form a triangle
sides are 7, 7, and 4 feet, 4 feet being the distance
n the feet of the braced legs. Assuming three pulleys
ı both upper and lower block and the weight of a 12-
ngth of 36-inch cast-iron pipe as 3 tons, what are the
ɜ in the tripod legs if the rope leading to the hoisting
ies in the plane of the braced legs and bisects the angle
. by them?

**er Legs.** — Another form of
much used in hoisting and
erring loads is illustrated in
5.

e $AB$ and $BC$ represent stiff
chored at $A$ and $C$ so as to
ı hinge-like motion about the line $AC$, and fastened
er at $B$.

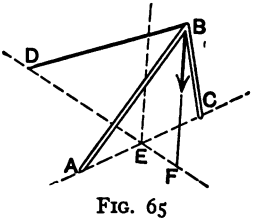

FIG. 65

se sheer legs are supported by a backstay, $BD$, so
d at $D$ that it may be lengthened and shortened,

thus changing the "reach" $EF$. This backstay may be so arranged that the projection of $B$ may fall between $E$ and $D$; then the backstay must take compression.

EXERCISE 115. Compute the stresses in the legs and backstay of the sheers erected at Sparrow's Point, Md., when carrying a load of 100 tons with a reach of 35 feet. The legs are each 116 feet long and their heels are 45 feet apart. The backstay meets the horizontal plane thru the heels 63.5 feet from the line joining them.

EXERCISE 116. Solve Ex. 115 by means of a plan and elevation.

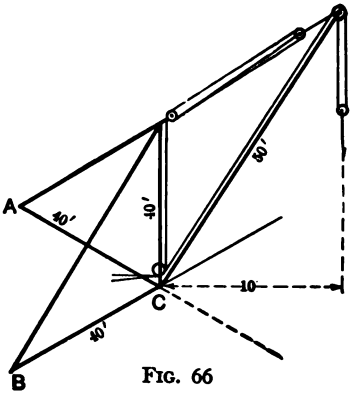

FIG. 66

EXERCISE 117. If the contractors' derrick,* illustrated in Fig. 66, carries a load of 2 tons, what weights must be placed at $A$ and $B$, what is the vertical pressure on the ground at $C$, and what horizontal forces must be applied to $AC$ and $BC$ to preserve equilibrium when (a) the jib lies in the position shown, (b) when it lies in the plane determined by the post and $AC$? ($\angle ACB = 90°$).

EXERCISE 118. What are the stresses in all members under the conditions assumed in Ex. 117?

### The Steam Shovel and the Dipper Dredge

EXERCISE 119. In the dipper dredge illustrated in Fig. 67, assume 5000 pounds as the greatest tension which the hoisting chain can stand and that the resistance to the motion of the dipper is horizontal.

* For photographs of derricks, steam shovels, etc., see advertisements in Engineering News, The Contractor, etc.

(a) Show the dipper and handle as a free body and find graphically all forces acting on it.

(b) Show the boom as a free body and find graphically all forces on it.

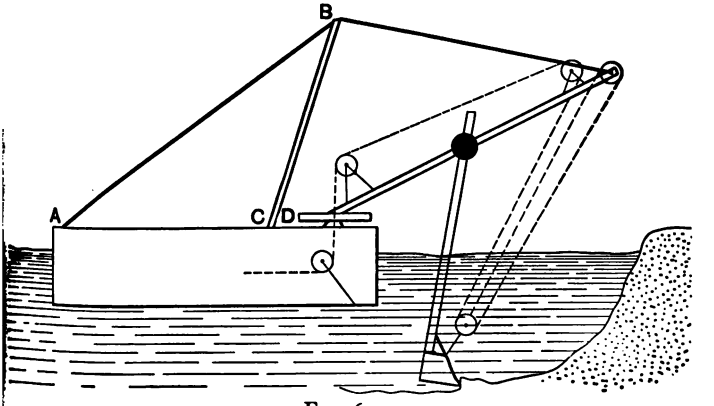


FIG. 67

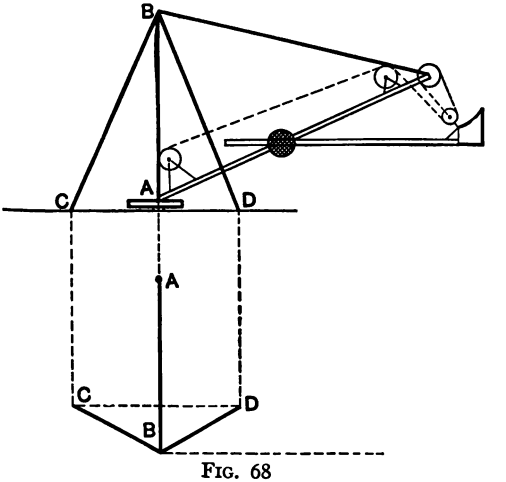


FIG. 68

EXERCISE 120. In Fig. 68 assume a load of 2000 pounds about to be discharged and find graphically the stresses in the shear legs and backstay supporting the end of the boom.

# CHAPTER V

# EQUILIBRIUM OF PERFECTLY FLEXIBLE ROPES AND CABLES

## SECTION XIV

### THE FUNICULAR POLYGON

THE position assumed by a weightless rope supporting given forces is illustrated in Fig. 69. Here the given

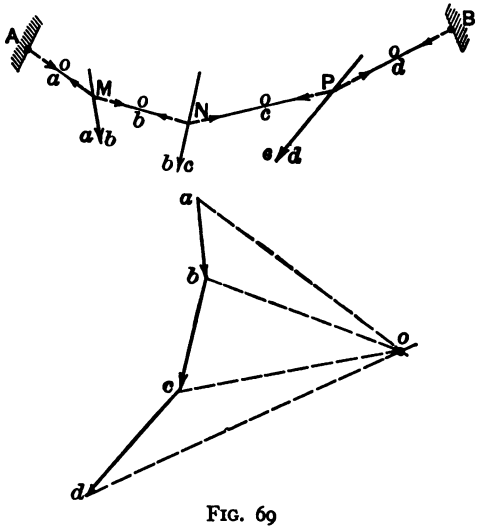


FIG. 69

forces are *ab*, *bc*, and *cd*; *o* is any pole, and *oa*, *ob*, *oc*, and *od* are the rays parallel to which are drawn the corresponding segments of the rope sustaining the given loads.

Any point on *oa*, such as *A*, and any point on *od*, such as *B*, may be assumed as fixed points to which the ends of the rope are fastened.

Note that three forces act at *M*, the knot at which the load *ab* is fastened to the segments of the rope *oa* and *ob*. The tensions in the ropes *oa* and *ob* are shown graphically in the lower diagram by the lengths of the rays *oa* and *bo* (note carefully their directions). These forces are in equilibrium and their triangle closes. The same follows for the knots *N* and *P*.

EXERCISE 121. If a weightless strut connects *A* and *B* (Fig. 69) and the external supporting force at *B* is vertical, find the magnitude of this supporting force, the compression in the strut and the magnitude and direction of the supporting force at *A*.

What forces at *A* and *B* (Fig. 69) would support the rope if the strut joining *A* and *B* were removed?

**To Find the Locus of the Poles of All Funicular Polygons Passing Thru Two Given Points.** — In Fig. 70 let *ab*, *bc*, and *cd* be the given forces, and *A* and *B* the two points thru which a funicular polygon, for this loading, should be passed.

Assume the line of action of some force *de* passing thru *B* and then find, in the usual manner, the magnitude of this force and the magnitude and direction of another force passing thru *A* so that these forces shall be in equilibrium with the given set of forces.

This may be done by assuming any pole *o* and constructing the corresponding funicular polygon which passes thru *A* but not necessarily thru *B*. As the forces, *ab*, *bc*, *cd*, *de*, and *ea* are now in equilibrium a change in

the position of the pole $o$ will change the funicular polygon but can in no way alter these forces. Thus, if thru $e$ a line $MN$ is drawn parallel to $AB$ it will be the new

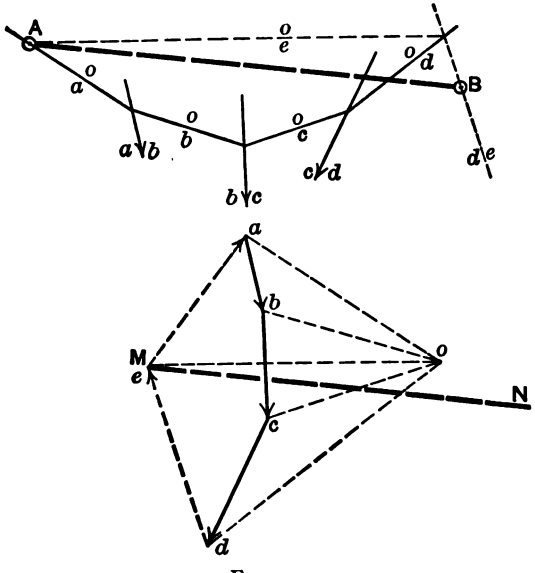


FIG. 70

ray $oe$ for a funicular polygon whose corresponding side is **AB**, and **any point on MN used as a pole will yield a funicular polygon passing thru A and B.**

EXERCISE 122. Solve the above problem if the lines of action of the assumed forces $de$ and $ea$ are taken mutually parallel.

EXERCISE 123. Solve the above problem if the point $B$ lies between the forces $bc$ and $cd$ and the side $oc$ of the funicular polygon is to pass thru it. (Solve in the same manner as above, neglecting the force $cd$.)

EXERCISE 124. In Fig. 70 assume in addition to $A$ and $B$ another point $C$ between forces $ab$ and $bc$ and then find the pole of the funicular polygon passing thru $A$, $B$, and $C$.

EXERCISE 125. Assume three vertical forces and any three points and plot four weightless rods, freely hinged at their ends, which will support the assumed loads and three of which pass thru the assumed points.

If the ends of the last rods are joined by a tie, find the vertical reactions at these ends and the tension in the tie.

## SECTION XV

### THE PARABOLA

**A Weightless Cable Sustaining a Uniformly Distributed Load.** — In Fig. 71 assume the cable and the verti-

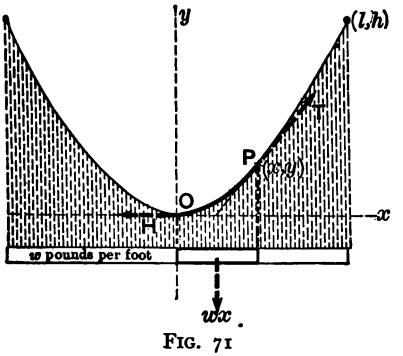


FIG. 71

cal suspension rods to be weightless so that the whole load carried by the cable is $w$ pounds per horizontal foot as indicated. This load must be understood to be divided into innumerable, very short portions, each suspended by its own rod from the cable.

Consider that portion of the figure drawn in heavy lines as a free body. The only forces acting upon it are the

forces $H$ and $T$ tangent to the cable at the points $O$ and $P$, and the weight of the load $wx$ whose center of gravity lies somewhere on the line parallel to the $Y$-axis and distant from it by $\frac{x}{2}$. As the lines of action of these three forces must pass thru the same point, the prolongation of $T$ must pass thru the point $\left(\frac{x}{2}, O\right)$, for $O$ is assumed at the lowest point in the cable where its tangent is evidently horizontal.

From Fig. 71 it follows that

$$\frac{dy}{dx} = \frac{y}{\frac{x}{2}},$$

for any point in the cable, whence, by integration,

$$y = cx^2,$$

and as $x = l$ when $y = h$, $c = \frac{h}{l^2}$,

or

$$y = \frac{h}{l^2}x^2$$

is the equation of the curve assumed by the cable.

This is the equation of a parabola tangent to the $x$-axis at $O$ and having the $y$-axis as axis.

From the triangle of forces showing the relation between the forces $H$, $T$, and $wx$, the slope of $T$ is $\frac{wx}{H}$, but as the slope of $T$ is $\frac{dy}{dx}$,

we have

$$\frac{dy}{dx} = \frac{wx}{H}.$$

As $H$ is constant (why? integration yields

$$y = \frac{wx^2}{2H} + c$$

and as $y = 0$ when $x = 0$, [illegible]

so that
$$y = \frac{wx^2}{2H}$$

is another form of the equation of the curve assumed by the cable.

The two equations $y = \frac{wx^2}{2H}$ and $y = \frac{ax^2}{p}$ represent the same curve referred to the same axes; thus their parameters must be the same.

or
$$\frac{w}{2H} = \frac{a}{p}$$

whence
$$H = \frac{wp}{2a}$$

EXERCISE 126. Find the horizontal tension in the cable by considering one-half of the cable in Fig. [illegible] as a free body; use no calculus.

EXERCISE 127. Find the tension in the cable at the points $(x, y)$ and $(l, h)$. Use no calculus or formulas.

EXERCISE 128. The span of the cables of a suspension bridge is 800 feet and their dip is [illegible] feet; assume the weight of load, platform, cables, and suspension rods as equivalent to 4500 pounds per horizontal foot. Find the greatest and least tension in each of the two cables.

EXERCISE 129. The load on a simple parabolic arch, 200-foot span, 20-foot rise, is 360 tons; determine the greatest and least thrusts on the arch.

**A Uniform Cable Sustaining Its Own Weight Only** is evidently not loaded at a constant rate per horizontal foot.

Nevertheless, when a cable sags but slightly and its points of suspension are on the same level, the length of the cable is very nearly equal to the distance between the points of suspension and very approximate results are obtained by assuming the load uniformly distributed per horizontal foot.

EXERCISE 130. An iron telegraph wire (No. 12, 0.1 inch in diameter, 3.15 pounds per 100 feet) is stretched between poles 240 feet apart. What is the least sag allowable if the wire can stand a tension of 60,000 pounds per square inch?

EXERCISE 131. A wire is just to touch a certain imaginary horizontal line and is to be fastened to two vertical poles at $h$ and $h'$ feet above this horizontal line. The poles are $l$ feet apart. How far from the second pole will the wire reach its lowest point?

EXERCISE 132. Find the length of the wire in Ex. 131.

## SECTION XVI

### THE CATENARY

The **accurate solution** of problems such as the above is necessarily more complicated.

In Fig. 72 consider the curve in which a uniform flexible cable hangs when referred to the axes of $x$ and $y$, as shown.

The portion of the cable between the points (0, 0) and $(x, y)$ is in equilibrium under the forces $H$, $T$, and $ws$, where $w$ is the weight of the cable per foot of length and $s$ is the length of cable between (0, 0) and $(x, y)$.

Evidently $H$ is a constant force, independent of $x$ or $y$. Let it be equal to $mw$, so that $m$ is the length of that portion of the cable whose weight equals $H$.

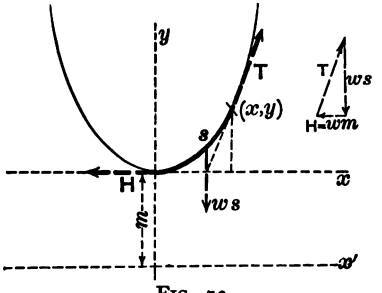


FIG. 72

Thus we assume that

$$H = wm.$$

The triangle of forces shows that

$$T = w\sqrt{s^2 + m^2},$$

and that the slope of the tangent at $(x, y)$ equals $\frac{ws}{wm} = \frac{s}{m}$. The slope of the curve at $(x, y)$ also equals

$$\frac{dy}{dx} = \frac{\sqrt{1 - \left(\frac{dx}{ds}\right)^2}}{\frac{dx}{ds}}$$

expressed in terms of $x$ and $s$.

Thus, the differential equation of the curve expressed in terms of $x$ and $s$ is

$$\frac{\sqrt{1 - \left(\frac{dx}{ds}\right)^2}}{\frac{dx}{ds}} = \frac{s}{m},$$

whence

$$dx = \frac{m\,ds}{\sqrt{m^2 + s^2}},$$

or $$\frac{x}{m} = \log_e \left\{ \sqrt{1 + \frac{s^2}{m^2}} + \frac{s}{m} \right\} + (c_1 = o).$$

Changing this to the exponential form and solving for $s$ we obtain from

$$\sqrt{1 + \frac{s^2}{m^2}} + \frac{s}{m} = \epsilon^{\frac{x}{m}},$$

$$s = \frac{m}{2}\left\{ \epsilon^{\frac{x}{m}} - \epsilon^{-\frac{x}{m}} \right\}.$$

To obtain the equation of the curve in terms of $x$ and $y$ note that

$$\frac{dy}{dx} = \frac{s}{m} = \frac{1}{2}\left\{ \epsilon^{\frac{x}{m}} - \epsilon^{-\frac{x}{m}} \right\},$$

whence $$y = \frac{m}{2}\left\{ \epsilon^{\frac{x}{m}} + \epsilon^{-\frac{x}{m}} \right\} + (c_2 = -m),$$

or $$y + m = \frac{m}{2}\left\{ \epsilon^{\frac{x}{m}} + \epsilon^{-\frac{x}{m}} \right\}.$$

The student who is familiar with the use of hyperbolic functions will perceive that the above solution may be condensed as follows:

From $dx = \dfrac{m\,ds}{\sqrt{m^2 + s^2}}$, obtained as above,

we have $$x = m \sinh^{-1} \frac{s}{m} + (c = o).$$

Whence $$\frac{s}{m} = \sinh \frac{x}{m}.$$

From Fig. 72 we see that $\frac{s}{m} = \frac{dy}{dx}$,

so that $\frac{dy}{dx} = \sinh \frac{x}{m}$,

or $y = m \cosh \frac{x}{m} + c.$

But as $x = 0$ when $y = 0$, $c = -m$,

and $y + m = m \cosh \frac{x}{m}.$

If we now refer the curve to a new axis of $x$, $x'$, $m$ units below the old axis of $x$ its equation becomes

$$y' = m \cosh \frac{x}{m}.$$

EXERCISE 133. Express $x$ in terms of $y'$ and $s$.

EXERCISE 134. Find the tension, $T$, Fig. 72, in terms of $w$, $m$, and $s$; also in terms of $w$, $m$, and $y$.

EXERCISE 135. Show that $(y + m)^2 = y'^2 = m^2 + s^2$.

The above calculations develop the following important facts concerning the **catenary**, the curve in which a uniform, flexible cable (supporting only its own weight and supported at two points only) hangs.

If the horizontal tension at the lowest point of the cable is $H = wm$,

then the tension at any other point $(x, y')$, Fig. 73, is

$$T = wy'.$$

The length, $s$, of the arc of the catenary from $(0, m)$ to $(x, y')$ is given by the equation

$$y'^2 = s^2 + m^2,$$

and the horizontal distance between $(0, m)$ and $(x, y')$ is

$$x = m \log_e \frac{y' + s}{m}.$$

In Fig. 73 the heavy line represents a flexible cable passing over two smooth pins of infinitesimal diameter at $(0, m)$ and $(x, y')$; if the portion $s$ hangs in a catenary and the overhang at $(0, m)$ is $m$ and at $(x, y')$ is $y'$, then this cable is in equilibrium supported only by the two pins as shown.

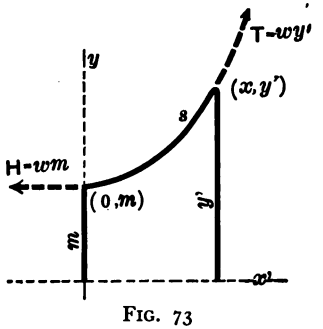


FIG. 73

EXERCISE 136. A 100-foot steel tape is supported at its ends by forces of 1.7 and 2.0 pounds, but otherwise the ends are free to assume any direction consistent with equilibrium. This tape weighs 0.005 pound per foot. Find the horizontal and vertical distances between the points of support.

EXERCISE 137. What is the distance between two points joined by a steel tape 100 feet long, weighing 0.005 pound per foot, if the tension handles at each end register 10 pounds? How much does the tape sag?

# CHAPTER VI

## FRICTION — APPLICATIONS TO MACHINES

A **machine** is a structure so designed that some of its members may be in relative motion. These motions form the subject of kinematics of mechanism.

The members of machines transmit forces and as these forces may be displaced **machines may transmit energy.**

As the motions of the various members of machines may be very complicated (as their **accelerations** may vary from instant to instant) and as all members necessarily have **mass,** computations involving all **forces** acting upon and transmitted by the members of machines may therefore become very complicated.

In the following sections **machines** will only be **considered from the standpoint of statics.** The machines will be regarded as about to move or moving very slowly (without acceleration) or better still the **members** of the **machines** will be regarded as massless so that inertia may be neglected.

Inertia forces resulting from the accelerations of the members of machines form the subject of kinetics of machinery.

**Friction.** — As relative motion between members of machines is always possible **friction** cannot be neglected in the study of the forces transmitted by these members as was done in the consideration of structures in which the members are relatively at rest.

## SECTION XVII

### SLIDING FRICTION

Whenever two bodies tend to slide over one another under the action of external forces and these bodies are pressed together a tangential force at their surfaces of contact always manifests itself. This tangential force, called a **frictional force** or simply **friction,** always opposes the motion (or tendency to move) of the body upon which it acts.

To fix our ideas conceive a body, weighing $W$ pounds, resting upon a horizontal plane, Fig. 74 (a). It is in equilibrium under the action of two forces, its weight $W$, and the reaction of the plane $R$. These forces have the same vertical line of action passing thru the center of gravity of the body.

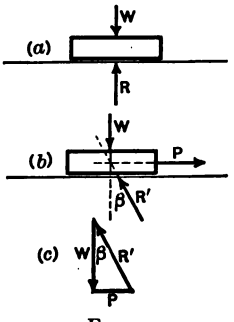

FIG. 74

If we now apply a horizontal force $P$ to this body, Fig. 74 (b), equilibrium may still persist but the reaction of the plane must now be inclined to the vertical at an angle $\beta$ such that the triangle of forces, Fig. 74 (c), may close.

Under these conditions the component of $R'$ normal to the surface of contact, say $N$, measures the pressure between these surfaces; the tangential component of $R'$, say $F$, is the frictional force called into play to prevent the motion which $P$ tends to impart to the body.

Assume $P$ to gradually increase; then if the body is to remain at rest, $\beta$ and $R'$ must increase so that the

tangential component of $R'$ (i.e., $F$) may continue to equal the increasing value of $P$.

Experience shows that provided $P$ does not exceed a certain limiting value, $F$ always remains equal to it. When the limiting value is reached $F$ attains its greatest value $F_s$ and the body is about to move. The slightest increase in $P$ now causes motion and this motion may be maintained with constant velocity by a force $P$ considerably less than that required to start the body. The frictional force to be overcome during motion is less than $F_s$; let it be denoted by $F_k$.

From the above it follows that problems involving sliding friction between bodies relatively at rest are indeterminate unless motion is about to occur, for the magnitude of the friction force may lie anywhere between zero and its limiting value $F_s$.

The **coefficient of static friction** $\mu_s$ is defined by the ratio

$$\frac{\text{limiting static friction}}{\text{normal pressure between surfaces in contact}} = \frac{F_s}{N},$$

and the **coefficient of kinetic friction** $\mu_k$ is defined by the ratio

$$\frac{\text{kinetic friction}}{\text{normal pressure between surfaces in contact}} = \frac{F_k}{N}.$$

Assume the angle $\beta$ to equal $\phi_s$ when the body in Fig. 74 is about to move and to equal $\phi_k$ when the body is in motion; then

$\phi_s$ is called the **angle of static friction,**

and $\phi_k$ is called the **angle of kinetic friction.**

In either case it is the greatest inclination between the common normal of the surfaces between which friction

occurs and the line of action of the total force transmitted from either surface to the other;

$$\mu_s = \tan \phi_s \quad \text{and} \quad \mu_k = \tan \phi_k.$$

**Coefficient of Sliding Friction.** — For the actual values of the coefficients of statical and kinetic friction as found by experiment the reader is referred to hand-books of engineering such as "*Hütte*" *des Ingenieurs Taschenbuch*, 1908, Vol. I, page 226 et seq., or *Kent's Mechanical Engineer's Pocket-Book*, 1910, page 1194 et seq.

In 1781 Coulomb published, under the title "Memoire sur la theorie des machines simples," the results of a series of experiments made by him on the friction of plane surfaces. These experiments were repeated and extended by Rennie* in 1829 and Morin† (1830–1834); their investigations led them to confirm in the main the results of Coulomb which are usually stated as follows:

The coefficient of statical friction is (1) independent of the normal pressure, (2) independent of the area of the surfaces in contact. The coefficient of kinetic friction is independent of the relative velocity of the surfaces and is always less than the corresponding statical coefficient.

That these empirical laws are correct for the conditions under which they were investigated is no doubt true. But the greatest pressure used by Morin in his experiments was about 100 pounds per square inch and his least pressure about three-fourths pound per square inch. Further investigations have shown that just above a pressure of 100 pounds per square inch deviations from Coulomb's laws begin and for pressures of less than three-

* Rennie: Phil. Trans. —, 143. 1829.

† Morin: Institut de France, Recueil des Savans Étrangers, IV, V.

ɔurths pound per square inch $\mu_s$ is found to increase .s the pressure decreases.

Again, the highest velocity used in Morin's investigations was about 10 feet per second, while for velocities greater than this $\mu_k$ diminishes with increasing velocity, as has been shown by Poirée * and by Bochet † and more recently by Galton.‡ Jenkin § showed that for relative velocities of 0.01 to 0.0002 feet per second kinetic friction gradually increases as the velocity diminishes so that there is probably a gradual transition between static and kinetic friction.

It should be remembered that the coefficient of sliding friction depends for its value upon (1) the material of the rubbing bodies, (2) the condition of their surfaces, (3) the lubricant and method of lubrication, (4) the normal pressure between the surfaces in contact, (5) the relative velocity of these surfaces, (6) the temperature.

Any value of $\mu_s$ or $\mu_k$ should thus only be used for conditions similar to those under which it was obtained. Too much credence should never be attached to these; independent experiments are usually necessary when accuracy is required.

The reader is referred to Thurston's books, *Friction and Lubrication* (1879) and *Friction and Lost Work in Machinery and Millwork* (1889) and for more recent references to the *Engineering Index* and *Engineering Index Annual.*

* Poirée: Mémoires de la Société des Ingénieurs Civils —, 113. 1852.
† Bochet: Annales des Mines (5) **19**, 27. 1861.
Comptes Rendus **46**, 802. 1858.
Comptes Rendus **51**, 974. 1860.
‡ Galton: Proc. Inst. Mech. Eng. —, 476, 590. 1878.
§ Jenkin: Phil. Trans. **167**, 509. 1877.

## Applications Involving Sliding Friction

EXERCISE 138. Show that the slope of a plane upon which a body otherwise unsupported is just on the point of sliding down equals the coefficient of statical friction between the body and plane.

As an **application** of the **angle of friction** to the solution of **problems involving sliding friction** let us find the **tractive force required to start a body** weighing $W$ pounds along a horizontal plane, the coefficient of static friction being $\mu = \tan \phi$ where $\phi$ is the angle of friction.

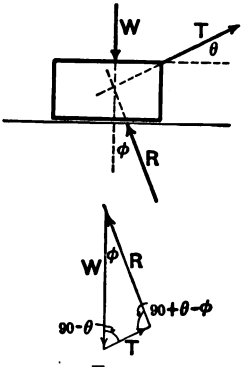

FIG. 75

As in Fig. 75, assume the inclination of the tractive force, $T$, to the horizon to be $\theta$. Only three forces act upon the body considered, i.e., $W$, $T$, and $R$ the reaction of the plane. Therefore, the line of action of $R$ must pass thru the intersection of the lines of action of $T$ and $W$ and **it must make an angle $\phi$ with the normal to the surface of contact.** The corresponding triangle of forces with the magnitudes of its angles is shown in Fig. 75, from which by the law of sines we obtain

$$T = \frac{W \sin \phi}{\sin (90 + \theta - \phi)} = \frac{W \sin \phi}{\cos (\theta - \phi)}.$$

EXERCISE 139. Solve the above problem by means of $\Sigma H = 0$ and $\Sigma V = 0$.

EXERCISE 140. Find analytically the minimum value of $T$ and the corresponding "angle of traction" $\theta$.

EXERCISE 141. Construct several triangles of forces such

as shown in Fig. 75 for various values of $\theta$ and from this diagram find graphically the minimum tractive force and the most advantageous value of $\theta$.

EXERCISE 142. Solve the above problem if $T$ is a push instead of a pull.

EXERCISE 143. To what do the above results reduce if the plane is smooth.

EXERCISE 144. Find the minimum force required to drag a body up an inclined plane (slope = tan $\alpha$), the coefficient of kinetic friction being tan $\phi$.

EXERCISE 145. From this minimum tractive force find the general value if the tractive force makes an angle $\theta$ with the plane.

EXERCISE 146. What horizontal force is required to push a body up an inclined plane?

EXERCISE 147. What force parallel to the plane is required to push a body up an inclined plane?

EXERCISE 148. Solve the general case for a body being dragged down a plane.

**The Wedge.** — Let it be required to find the force $P$, Fig. 76, necessary to (a) just start the wedge towards the left, (b) just prevent the wedge from moving towards the right. In the solution we shall assume the coefficient of friction, or, what amounts to the same thing, the corresponding angles of friction, to be known and different for each pair of rubbing surfaces.

In Fig. 76 (a) each movable part of the machine is shown as a free body when the wedge is about to start towards the left under the action of the force $P$. Note carefully the directions of the various reactions of the surfaces of contact. As friction invariably opposes the motions about to occur **the reactions are inclined to the normal at the angles of friction upon that side of the**

normal which gives to the tangential component the reactions directions opposing the impending rel motion of the body upon which they act.

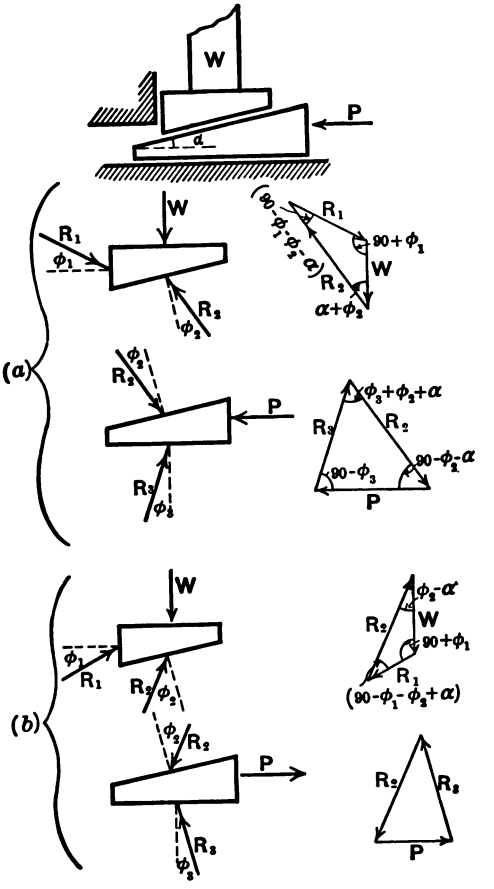


FIG. 76

The actual lines of action of the reactions are not in cated; in fact these are indeterminate; only their dir

tions are known, and as in each case only three forces act we know that their lines of action must pass thru one point.

From the corresponding triangles of forces the unknown reactions and $P$, the motive force, may be determined graphically, or, by an application of the law of sines, we may obtain

$$R_2 = \frac{W \cos \phi_1}{\cos (\phi_1 + \phi_2 + \alpha)},$$

and then

$$P = \frac{R_2 \sin (\phi_3 + \phi_2 + \alpha)}{\cos \phi_3} = \frac{W \cos \phi_1 \sin (\phi_3 + \phi_2 + \alpha)}{\cos \phi_3 \cos (\phi_1 + \phi_2 + \alpha)}.$$

In Fig. 76 (b) are shown the free bodies and their triangles of forces when the wedge is about to move towards the right. Now $W$ becomes the motive force and the second triangle of forces shows that $W$, with the assumed values of $\phi$, is unable to start the wedge outward for **P must help it** to the extent shown if the wedge is about to move towards the right. This machine is therefore said to be self-locking. How do you explain the direction of the force $R_1$ in the triangle of forces?

From the triangles of forces we obtain

$$R_2 = \frac{W \cos \phi_1}{\cos (\phi_1 + \phi_2 - \alpha)},$$

and

$$P = \frac{W \cos \phi_1 \sin (\phi_2 + \phi_3 - \alpha)}{\cos \phi_3 \cos (\phi_1 + \phi_2 - \alpha)}.$$

If the coefficients of friction are all equal we obtain as answer to (a) $P = W \tan (\alpha + 2\phi)$, and for (b) $P = W \tan (2\phi - \alpha)$.

EXERCISE 149. If the coefficients in the above problem are all equal to $\mu$, for what values of $\alpha$ is the machine self-locking?

EXERCISE 150. Solve the above problem of the wedge if the force $P$ (Fig. 76) is inclined to the horizontal at an angle $\beta$.

EXERCISE 151. In Fig. 77 is shown an arrangement for adjusting the rollers of a rolling mill. What force must be exerted at $P$ so as to raise the bearing upon which a load of 1600 pounds rests? Assume the angle

$$\alpha = 5^\circ, \quad \mu = 0.16, \quad \mu_1 = 0.18, \quad \text{and} \quad \mu_2 = 0.15.$$

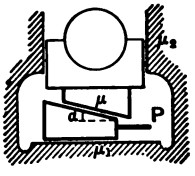


FIG. 77

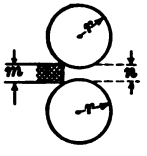

FIG. 78

EXERCISE 152. What would be the result in Ex. 151 if all surfaces were perfectly smooth?

EXERCISE 153. In Fig. 78 are shown the rolls of a rolling mill and a bar about to enter the mill. Show that if $m - n \leqq 2r(1 - \cos\phi)$, where $\phi$ is the angle of friction between the rolls and the bar, the bar will enter the rolls unassisted.

EXERCISE 154. What normal pressure does a wedge (angle $\alpha$) exert when used for splitting if driven by a force $P$ and if the angle of friction is $\phi$?

**Keys and Cotters.** — To find the relation between the force $P$ applied to a cotter, Fig. 79, and the total force, $Q$, exerted by the cotter in a direction at right angles to $P$, make a free body of the cotter as shown. Then the

corresponding triangle of forces yields as the vertical components, $Q$, of both $R_1$ and $R_2$,

$$Q = m \cot \phi = n \cot (\alpha + \phi),$$

whence

$$P = m + n = Q [\tan \phi + \tan (\alpha + \phi)].$$

EXERCISE 155. What is the least value of $P$ which will hold the cotter in position against a given compression $Q$?

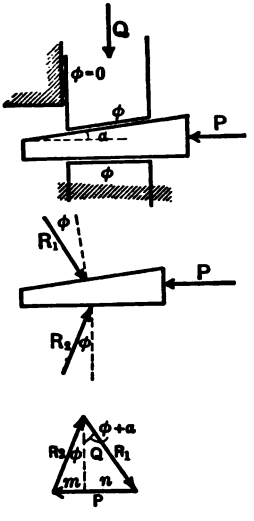


FIG. 79

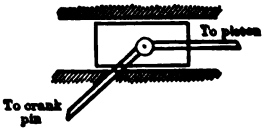


FIG. 80

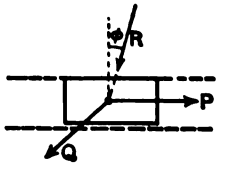


FIG. 81

EXERCISE 156. What limiting angle $\alpha$ will allow the cotter to remain in position if $P$ is reduced to zero?

If $\mu = 0.08$ for iron on iron to what taper does this correspond?

**Guides.** — As another application of the angle of friction to problems involving sliding friction consider the behavior of the **crosshead of an engine** under the

action of the forces transmitted to it thru the piston-rod, connecting-rod, and crosshead guides. (Fig. 80.)

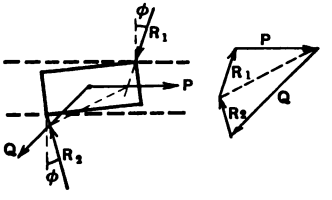


FIG. 82

Assume the crosshead to be moving to the right, then Figs. 81, 82, 83, and 84 represent the four conditions to be investigated.

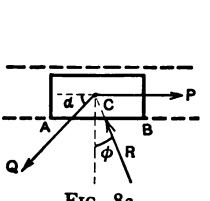


FIG. 83

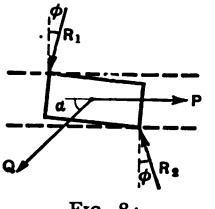


FIG. 84

Fig. 81 shows that the triangle of forces necessary for equilibrium could not be closed with the assumed direction of the forces. Thus this condition is impossible.

EXERCISE 157. Assume the magnitude of $P$ in Fig. 81 and complete the triangle of forces. Note the direction of $R$ necessary for equilibrium.

The condition illustrated in Fig. 82, with the construction for four forces in equilibrium there shown discloses that here again one of the reactions of the guides ($R_1$) must have an impossible direction and hence this case also is impossible.

Fig. 83 illustrates a **possible condition.** Here the three forces pass thru a point and their triangle (not shown) will close. Try it. Of course the lower **guide cannot furnish the force R unless the crosshead is sufficiently long** relatively to its width to bring the point $C$ within the length $AB$.

EXERCISE 158. Show that if the length of the crosshead is $l$ and its height is $h$ the condition illustrated in Fig. 83 is only possible if $\frac{l}{h} \geqq \mu$, the coefficient of friction.

EXERCISE 159. Show that the following relation exists between $P$ and $Q$ under the condition of Ex. 158:

$$P \cos \phi = Q \cos (\alpha - \phi) = Q (\cos \alpha + \mu \sin \alpha) \cos \phi.$$

Fig. 84 illustrates the condition, provided the requirements of Ex. 158 are not fulfilled, namely, if $\frac{l}{h} < \mu$.

EXERCISE 160. Show that under the conditions assumed to exist in Fig. 84 $\left(\text{namely, } \frac{l}{h} < \mu\right)$,

$$P = Q \left(\cos \alpha + \mu^2 \frac{h}{l} \sin \alpha\right).$$

(This result is best obtained analytically.)

EXERCISE 161. In Fig. 84 assume the magnitudes of $P$, $\phi$, $l$, and $h$ and find $Q$ graphically.

Exs. 159 and 160 show that it is advantageous to make the length of the crosshead sufficiently great to insure the conditions illustrated in Fig. 83.

EXERCISE 162. A lift, Fig. 85, slides upon a rod 2 inches square. The extreme length between bearing surfaces is 1 foot. The resultant weight of lift and load (400 pounds) acts 2 feet from the center of the rod. If $\mu = 0.2$ show that

the "lifting" force applied 6 inches from the center of the rod is

(a) 100 pounds when the lift descends,

(b) 600 pounds when the lift ascends,

(c) 400 pounds when friction is neglected.

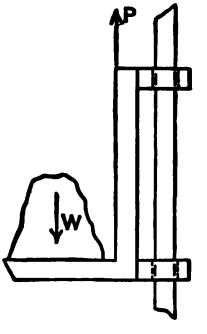


FIG. 85

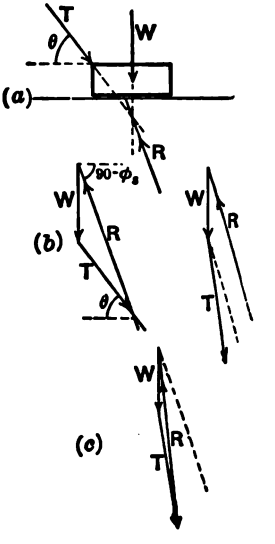


FIG. 86

## The Cone of Friction

If, in the solution of Ex. 142, the force $T$, Fig. 86 (a), is inclined at an angle $\theta \equiv 90 - \phi_s$, where $\phi_s$ is the static angle of friction, then it will be found that no matter what extent the force $T$ be increased, motion is impossible. For, as Fig. 86 (b) shows, the triangle of forces exhibiting the relations of the forces when motion is about to occur can no longer be closed when $\theta \equiv 90$. This does not mean that equilibrium cannot exist

Fig. 86 (c) shows, the limiting friction is no longer led into play to preserve rest. For a given weight, , and push, $T$, the necessary frictional force is deter- ined by the horizontal component of $R$ in Fig. 86 (c) nd the angle of friction is now less than $\phi_s$.

Under the condition $\theta \equiv 90 - \phi_s$ the body is said to e **jammed**, and under these conditions the **applied orces, no matter what their magnitudes may be, cannot roduce motion.**

It also follows that jamming will occur whenever the ne of action of $T$ lies within the conical surface having le intersection of the lines of tion of $T$ and $W$ as vertex d a semi-angle equal to $\phi_s$. is conical surface is called the **e of friction.**

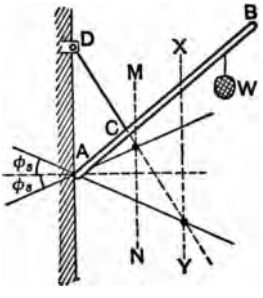


FIG. 87

Another application of the e of friction is illustrated in . 87. Here a rod $AB$ rests at gainst a vertical wall and is ened to $D$ by the string $CD$. f the resultant weight of the and its load lies between the verticals $MN$ and $XY$ motion can occur, for the limiting friction at $A$ is exceeded. When this resultant lies on either of the ticals $MN$ or $XY$ motion is impending, and if it lies hout, on either side, motion occurs.

EXERCISE 163. Show the forces acting upon the rod $AB$ g. 87) and draw the corresponding triangles of forces when line of action of the resultant load

a) lies between $MN$ and $XY$,

b) coincides with $MN$,

(c) coincides with $XY$,

(d) lies to the right of $XY$ (which way will the rod move?),

(e) lies to the left of $MN$ (which way will the rod move?).

EXERCISE 164. A ladder rests against a vertical wall and upon horizontal ground. Assume the coefficient of friction at both wall and ground and plot the limiting positions for the resultant load on the ladder consistent with equilibrium. Do this for several inclinations of the ladder.

Consider the case of **an arm arranged to slide upon a vertical rod** (Fig. 88). This arm will be acted upon by

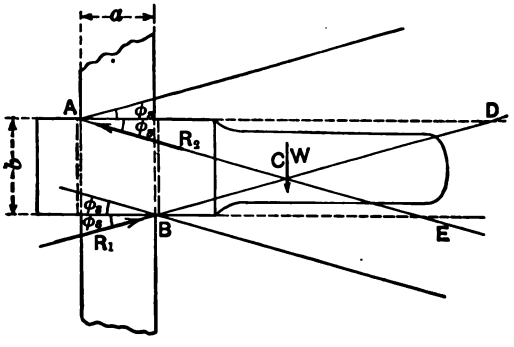


FIG. 88

three forces: a force at $A$; another at $B$, both being reactions of the rod upon the arm; and the resultant load, $W$, which we shall assume vertical.

The reactions at $A$ and $B$ must lie within the cones of friction, and if the line of action of the resultant load, $W$, passes thru $C$ the limiting friction at **both** $A$ and $B$ is called upon to prevent the arm from moving (slipping). Under these conditions the forces acting upon the arm are indicated by the arrows in Fig. 88.

EXERCISE 165. Explain the nature of the reactions when the line of action of the resultant load $W$ Fig. 88 passes thru the intersection (a) at $D$; b at $E$; c when the line of action of the resultant lies between $C$ and $B$.

In Fig. 88 note that the reactions at $A$ and $B$ are indeterminate whenever the resultant load $W$ does not pass thru $C$. Also the arm is jammed whenever this resultant lies to the right of $C$ and motion results if it lies to the left of $C$.

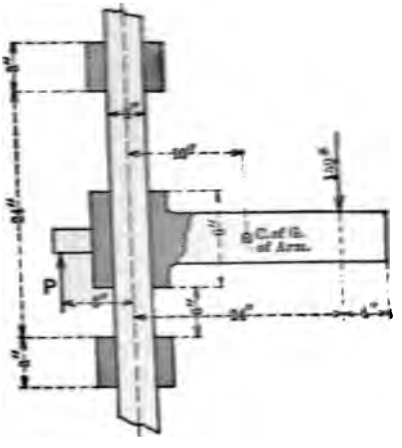


FIG. 89

EXERCISE 166. (a) Show that if the distance between the line of action of the resultant $W$ and the center line of the rod (Fig. 88) is equal to or greater than $\frac{b}{2\mu}$, where $\mu$ is the coefficient of friction and $b$ is as indicated in the figure, then motion is impossible.

(b) Should the dimension $b$ (Fig. 88) be large or small if the arm is designed to slide upon the rod?

EXERCISE 167. In the design of a loom the following

problem occurred. An arm is to be raised by a force $P$ (Fig. 89); the arm supports a force of 150 pounds as shown and weighs 30 pounds. The vertical shaft weighs 40 pounds. In order to reduce $P$ to its least value should the arm slide upon the shaft (which is stationary) or should the arm be pinned to the shaft, the sliding then to occur at the fixed bearings?

## SECTION XVIII

### EFFICIENCY OF MACHINES

If in any machine

$P$ = the driving force,

$Q$ = the useful resistance,

$F$ = any frictional force,

and $p$ represents the effective displacement of $P$,

$q$ represents the effective displacement of $Q$,

$f$ represents the effective displacement of $F$,

then

$Pp$ equals the total work supplied to the machine,

$Qq$ equals the useful work delivered by the machine,

and $Ff$ equals the work lost thru the action of the friction $F$.

The **efficiency, $\eta$, of a machine** is defined by the ratio

$$\eta = \frac{\text{output}}{\text{input}} = \frac{\text{useful work delivered by the machine}}{\text{total work supplied to the machine}},$$

thus

$$\eta = \frac{Qq}{Pp}.$$

Moreover, the principle of work furnishes the following equation

$$Pp - Qq - \Sigma Ff = 0,$$

under the assumption that the members of the machine are massless, so that no work is stored and restored as

kinetic energy by reason of any change in the velocities of the members of the machine.

If $P_0$ represents the **driving force** which would overcome the useful resistance, $Q$, **provided the machine were frictionless**, then

$$P_0 p - Qq - \Sigma(o) f = o,$$

and

$$\eta = \frac{Qq}{Pp} = \frac{P_0 p}{Pp} = \frac{P_0}{P}.$$

Hence, the efficiency of a machine may be found by dividing the driving force, which, applied to an ideal frictionless machine (otherwise identical with the original machine), would overcome the same useful resistance as a driving force $P$ applied to the actual machine, by $P$ itself.

EXERCISE 168. Show that the efficiency equals $\frac{Q}{Q_0}$, the actual useful resistance overcome divided by the useful resistance overcome when friction is neglected, with the result from the same driving force.

EXERCISE 169. Find the efficiency of the lift in Ex. 162 from the results there obtained, (a) when ascending, (b) when descending.

EXERCISE 170. Find the efficiency of the machine described in Ex. 151.

**Overhauling and Self-locking of Machines.** — A machine is said to **overhaul** if it runs backward whenever the driving force is removed. To illustrate, return to Ex. 162; here it was found that $P$ would have to equal 100 pounds when the lift is about to move downward; thus, if $P$ were reduced to zero the lift would fall (i.e., run backwards).

A **machine is** said to be **self-locking when on the removal of the driving force it remains jammed** in position under the action of the useful resistance and the frictional forces.

A self-locking machine cannot reverse unless the original driving force is reversed in direction so as to assist the original useful resistance, which now becomes a driving force, to overcome the frictional resistances.

Under these conditions the equation derived by means of the principle of work would read

$$Pp + Qq - \Sigma Ff = 0,$$

for not only does $Q$ become the driving force in the machine when reversed, but $P$, which now constitutes the "useful resistance," is also a "driving force."

The **efficiency** of a **reversed self-locking machine** is thus

$$\eta = \frac{-Pp}{Qq} = \frac{-P}{P_0} = -\frac{P}{P_0},$$

a **negative quantity,** where $P_0$ is the value of the force $P$ when friction is neglected and is opposite in direction to $P$.

In any **self-locking machine** when running forward we have

$$P_f p - Qq - \Sigma F_f f = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and when motion is reversed we have

$$Qq + P_r p - \Sigma F_r f = 0, \quad . \quad . \quad . \quad . \quad (2)$$

where the subscripts $f$ and $r$ refer to the values of $P$ and $F$ for forward and reversed motions respectively and the useful resistance is kept constant.

Subtracting equation (2) from equation (1) we find that

$$P_f p - P_r p - 2Qq = \Sigma F_f f - \Sigma F_r f \quad . \quad . \quad (3)$$

Moreover, since the forces acting upon the machine during forward motion are in general greater than those acting during reversed motion,

$$\Sigma F_f f > \Sigma F_r f \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It thus follows from equations (3) and (4) that

$$P_f p > 2Qq$$

in any self-locking machine, provided the friction during forward motion exceeds the friction during backward motion.

From the above the **efficiency** of a **self-locking machine during its forward motion**

$$= \frac{Qq}{P_f p} < \frac{Qq}{2Qq}, \text{ or } \frac{1}{2},$$

i.e., the **efficiency is less than 50%**, when $F_f > F_r$.

Self-locking machines cannot be advantageously used when energy is to be transmitted, but for other purposes (such as hoisting) they are not only convenient but give great security.

EXERCISE 171. What efficiency during forward motion results in the case of a machine, if the frictional resistances are assumed the same during forward and backward motions and if the efficiency for reversed motion is zero?

EXERCISE 172. Show that the *efficiency of a complex machine equals the product of the efficiencies of its individual parts.*

EXERCISE 173. From the result of Ex. 172 show that the efficiency of a machine must decrease with an increase in the number of its parts.

## SECTION XIX

### SCREWS

The discussion of screws used for transmitting force or energy involves sliding friction.

**Square-threaded Screws.** — In Fig. 90 assume the couple, formed by the forces $\frac{P}{2}$ at a distance $l$ from the axis of the screw, to be on the point of overcoming the useful resistance $Q$ acting along the axis of the screw.

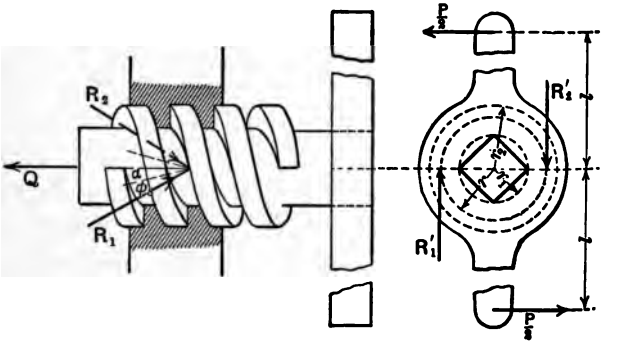


FIG. 90

Then the only other forces acting upon the screw are the reactions of the nut upon it. These reactions will be distributed over the left-hand helicoidal surfaces of the threads. The forces acting along any element of these surfaces (from a radius $r_1$ to a radius $r_2$) may, without serious error, be assumed concentrated on this element at a mean radius $r = \frac{r_1 + r_2}{2}$, so that all reactions may be assumed to act along a helix lying upon a cylinder of radius $r$.

The sum of the projections of these reactions, $\Sigma R'$, upon a plane normal to the axis must then balance the moment due to the two driving forces $\frac{P}{2}$, so that $r\Sigma R' = Pl$

or $$\Sigma R' = \frac{Pl}{r}.$$

Moreover, as Fig. 90 shows, the normal to the helicoidal surface is everywhere inclined to the axis of the screw at an angle $\alpha$, the pitch angle of the threads, and thus the reaction between nut and threads must be inclined at an angle of $\alpha + \phi_s$ to the axis. Thus the sum of the reactions themselves, $\Sigma R$, will be

$$\Sigma R = \frac{\Sigma R'}{\sin(\alpha + \phi_s)} = \frac{Pl}{r \sin(\alpha + \phi_s)},$$

and as the sum of the axial components of the reactions $R$ must equal $Q$, the only other axial force,

we have $$(\Sigma R)\cos(\alpha + \phi_s) = Q,$$

or $$P = \frac{Qr}{l}\tan(\alpha + \phi_s).$$

EXERCISE 174. What is the magnitude of $P$ when $Q$ is about to overcome $P$?

EXERCISE 175. What graphical solutions do the above formulas suggest?

EXERCISE 176. Under what conditions is a screw self-locking?

EXERCISE 177. What is the efficiency of a screw for

(a) forward motion?

(b) backward motion?

Explain the result of (a) when

(c) $\alpha = 0$,

(d) $\alpha = 90 - \phi$.

EXERCISE 178. In practice $\tan \alpha$ varies from 0.05 to 0.15. Show that in this case thorough lubrication is essential whenever a screw transmits work.

EXERCISE 179. Show that the work lost per minute in friction of screws during forward motion is

$$2\pi rnQ\{\tan(\alpha+\phi)-\tan\alpha\},$$

and when $\alpha$ is small that this approximates $2\pi rnQ\mu$.

EXERCISE 180. Show that $P=\dfrac{Qr(p+\mu\pi d)}{l(\pi d-\mu p)}$, where $p$ is the pitch and $d$ is the mean diameter of the screw, is equivalent to

$$P=Q\frac{r}{l}\tan(\alpha+\phi).$$

Which of these formulas is best adapted to numerical calculations?

EXERCISE 181. What value of $\alpha$ gives maximum efficiency for a given coefficient of friction?

**V-threaded Screws.** — As in Fig. 90 let $Pl$ be the moment of the driving couple and $Q$ the useful resistance.

In the v-threaded screw the angle between the normal to the helicoidal surface and the axis of the screw is not so easily determined as in the case of the square-threaded screw.

In Fig. 91 assume the cylinder (radius, $r$) to represent the mean cylinder of the screw threads. Let $B$ be any point on the mean helix of the surface of the thread bearing upon the nut, and $BA$ a tangent to this helix at $B$; then **$\alpha$ is the pitch angle.** Also let $BC$ be an element of the helicoid forming the surface of the screw; then **$\beta$ is the semi-angle of the thread,** and the plane $ABC$ is the tangent plane to the helicoid at $B$. The angle **$\gamma$ between**

the normal to this plane and the axis of the screw must now be determined in terms of $\alpha$ and $\beta$.

Fig. 92 shows the relation between this normal and its projections on the vertical planes thru $AB$ and $BC$ in Fig. 91. From Fig. 92 we obtain

$$\cos\gamma = \frac{1}{\sqrt{1 + \tan^2\alpha + \tan^2\beta}}.$$

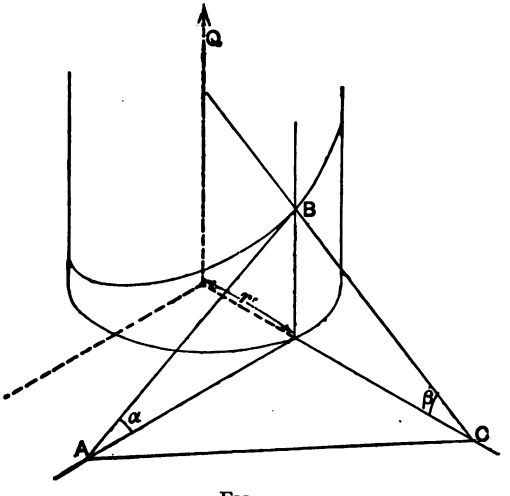


FIG. 91

At every point along the mean helix of contact (such as $B$) we thus find two forces acting: a normal force $N$ making an angle $\gamma$ with the axis of the screw and a tangential (frictional) force $\mu N$ making an angle $\alpha$ with a plane normal to the axis of the screw (acting in the direction $AB$ at $B$, Fig. 91).

As the sum of the forces along the axis of the screw must be zero, we have

$$Q - (\Sigma N)\cos\gamma + \mu(\Sigma N)\sin\alpha = 0. \quad . \quad . \quad (1)$$

Applying the principle of work to the screw as a who
we have for a displacement of one revolution,

$$2\pi Pl - (\Sigma N)(0) - \mu(\Sigma N)\, 2\pi r \sec\alpha - Q\, 2\pi r \tan\alpha = 0. \quad (:$$

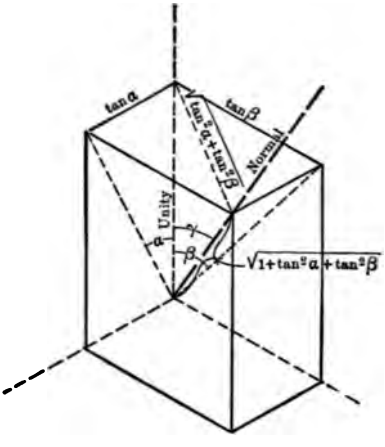


FIG. 92

Eliminating $(\Sigma N)$ from (1) and (2) there results

$$P = \frac{Qr}{l}\left\{\tan\alpha + \frac{\mu\sec\alpha}{\cos\gamma - \mu\sin\alpha}\right\}$$

$$= \frac{Qr}{l}\left\{\frac{\tan\alpha\cos\gamma + \mu(\sec\alpha - \tan\alpha\sin\alpha)}{\cos\gamma - \mu\sin\alpha}\right\},$$

or, dividing numerator and denominator by $\cos\gamma$,

$$P = \frac{Qr}{l}\,\frac{\tan\alpha + \dfrac{\mu(\cos\alpha)}{\cos\gamma}}{1 - \dfrac{\mu\sin\alpha}{\cos\gamma}},$$

which form suggests replacing $\dfrac{\mu\cos\alpha}{\cos\gamma}$ by, let us s:

$\phi'$, where $\phi'$ is a pseudo-friction-angle, so that

$$P = \frac{Qr}{l}\left\{\frac{\tan\alpha + \tan\phi'}{1 - \tan\alpha\tan\phi'}\right\},$$

$$P = \frac{Qr}{l}\tan(\alpha + \phi'),$$

ere, under our assumption,

$$\tan\phi' = \mu\cos\alpha\sqrt{1 + \tan^2\alpha + \tan^2\beta}.$$

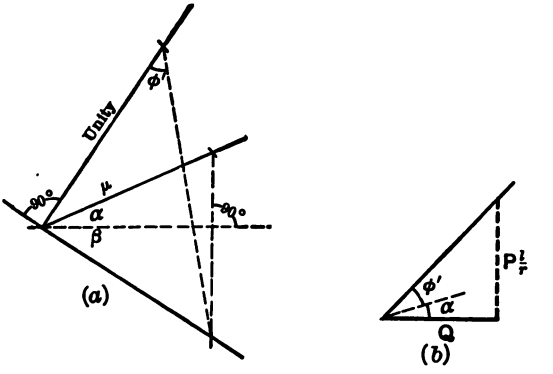


FIG. 93

, as is usually the case, $\alpha$ is small as compared to $\beta$,

$$\frac{1}{\cos\gamma} = \sqrt{1 + \tan^2\alpha + \tan^2\beta},$$

:h approximately

$$= \sqrt{1 + \tan^2\beta} = \sec\beta = \frac{1}{\cos\beta},$$

we have $\tan\phi' = \mu\dfrac{\cos\alpha}{\cos\beta}$, approximately.

EXERCISE 182. Show that the construction indicated in Fig. 93 (a) furnishes the approximate value of $\phi'$, and that the construction indicated in Fig. 93 (b) gives $P \frac{l}{r}$ for a v-threaded screw, for a given value of $Q$.

The formulas for the **efficiency** of **v-threaded screws** are the same as those for square-threaded screws provided the value of $\phi'$ is used instead of $\phi$, the actual angle of friction.

## SECTION XX

### PIVOTS

In the case of a pivot and its step bearing (Fig. 94) the elements of the surfaces in contact at which friction occurs are evidently at varying distances from the axis of rotation. The sum of the frictional forces acting at the elements of contact has thus no meaning. The frictional resistance must here be computed as a **resisting moment.**

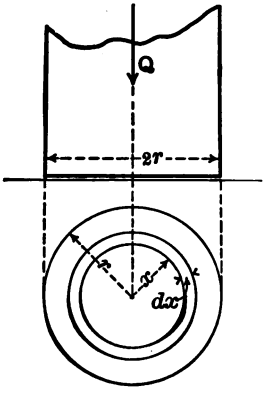


FIG. 94

In order to establish formulas for the frictional resisting moment various assumptions must be made. Only the simplest assumptions and the results based upon them will be considered. With our present experimental knowledge it is impossible to determine which assumptions lead to the most serviceable formulas.

The assumptions made in establishing the following formulas are

1. That the coefficient of friction is independent of the pressure and of the velocity at the point of contact; thus, $\mu$ is constant for any given pivot.

2. That the pressure over the bearing surface is either

(a) constant, or

(b) varies in such a manner as to allow of constant wear of the pivot or of the step in the direction of the thrust.

The assumption 2 *a* is sometimes said to correspond to the conditions which occur in new accurately fitted pivots, while the assumption 2 *b* is assumed to hold for worn pivots.

**Flat Pivots.**—Let $Q$ = the total thrust upon the pivot,

$r$ = the external radius of the pivot,

$\mu$ = the coefficient of sliding friction,

and $p$ = the normal pressure (force per unit area) over the bearing surface.

Consider now an annular element of the bearing surface whose area is

$$2\pi x\,dx \text{ (Fig. 94)}.$$

The normal force on this element is

$$(2\pi x\,dx)\,p,$$

the frictional force is $\mu\,(2\pi x\,dx)\,p$, and the frictional resisting moment is

$$\mu\,(2\pi x\,dx)\,px.$$

The total resisting moment for the whole pivot is

$$M = \int_0^r 2\pi x^2 p\mu\,dx = 2\pi\mu p \int_0^r x^2\,dx,$$

if we assume conditions 1 and 2 (a) stated above; for under these conditions $\mu$ and $p$ are constants.

Thus, $$M = \tfrac{2}{3}\pi\mu p r^3.$$

We must now determine $p$. The equation establishing vertical equilibrium of translation, namely,

$$Q = \int (2\pi x\, dx)\, p$$

yields $$p = \frac{Q}{\pi r^2}.$$

Hence, $$M = \tfrac{2}{3} r\mu Q.$$

Note that if $\mu Q$ be considered the total frictional force it may be considered to act at a mean radius of $\frac{2}{3} r$, under the assumptions made above.

EXERCISE 183. A flat pivot which presses its step with a force of 12,000 pounds is 6 inches in diameter and makes 80 revolutions per minute. If $\mu = 0.07$ find the horse-power lost in friction.

EXERCISE 184. Show that under the above assumption of force distribution the frictional force $\mu Q$ in the case of a collar pivot (Fig. 95) may be considered to act at a radius of $\frac{2}{3}\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}$.

**Conical and Hemispherical Pivots.** — In the derivation of the formula for conical pivots under the assumption of a constant normal pressure, $p$, over the bearing surface care should be taken of the fact that the frictional force on any element of the surface (Fig. 96) depends directly upon this normal pressure, while for vertical equilibrium of translation only the vertical component of the force due to $p$ should be considered.

EXERCISE 185. Show that the resisting moment for the conical pivot illustrated in Fig. 96 is $M = \frac{2}{3 \sin \alpha} \frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} \mu Q$, under the assumption of uniformly distributed force over the bearing surface.

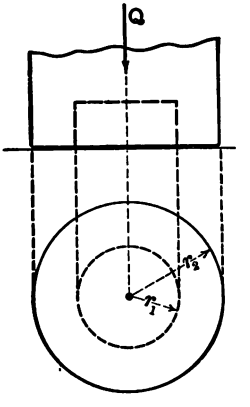


FIG. 95

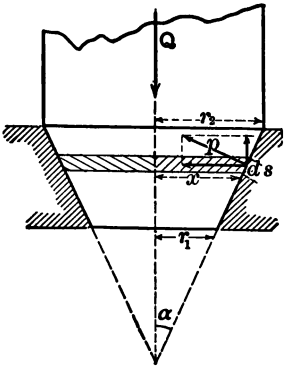


FIG. 96

If a conical pivot (Fig. 96) wears under running conditions it is reasonable to assume that the wear at all points of contact will be such as to assure continued fit between pivot and step. Thus under assumption 2 (b) we assume that the wear in the direction of the thrust is constant. This assumption, as will be seen, necessarily implies a variation in the magnitude of the normal pressure, $p$.

Wear depends upon the work wasted in friction at any point. This work equals $(\mu p dA)\ 2\pi x$ per revolution for any area $dA$. The energy expended per unit area of contact is thus $2\pi\mu p x$, and this energy causes wear normal to the surface of contact. This normal wear will

allow the pivot to move a distance proportional to $\frac{2\pi\mu px}{\sin\alpha}$ in the direction of the thrust.

Thus, to insure a continued fit of any pivot,

$$\frac{px}{\sin\alpha} \text{ must be constant,}$$

for we assume $\mu$ constant in these discussions.

In a conical pivot $\alpha$ is of course also constant so that here $px$ is constant.

Applying this condition to the conical pivot (Fig. 96) we have

$$M = \int_{r_1}^{r_2} (2\pi x\, ds)\, p\mu x = 2\pi\mu px \int_{r_1}^{r_2} x\, ds$$

$$= 2\pi\mu px \int_{r_1}^{r_2} x \frac{dx}{\sin\alpha} = \frac{\pi\mu px}{\sin\alpha}(r_2^2 - r_1^2).$$

To find $px$, we have

$$Q = \int_{r_1}^{r_2} p(2\pi x\, ds)\sin\alpha = 2\pi(px)\int_{r_1}^{r_2} dx = 2\pi(px)(r_2 - r_1),$$

or
$$(px) = \frac{Q}{2\pi(r_2 - r_1)},$$

so that
$$M = \frac{r_2 + r_1}{2\sin\alpha}\mu Q.$$

EXERCISE 186. (a) Show that the above formula reduces to $\frac{1}{2} r\mu Q$ for a flat pivot. (Compare with result obtained on page 130.)

(b) Plot the variation in pressure on the bearing surface of a flat and also of a collar pivot under the assumptions 1 and 2 (b) stated on page 129.

EXERCISE 187. Show that in a hemispherical pivot the frictional resisting moment is

(a) $\frac{2}{\pi} r\mu Q$, under the assumption of uniform axial wear,

(b) $\frac{\pi}{2} r\mu Q$, under the assumption of constant pressure,

(c) $r\mu Q$, under the assumption that the intensity of the normal pressure equals $k \cos\theta$, where the radius to the point of contact makes an angle $\theta$ with the vertical and $k$ is a constant to be determined from the equilibrium of the vertical components of the forces acting upon the pivot. Give the reasons upon which the assumptions made in (c) are based.

In each case $Q$ is the thrust on the pivot, $\mu$ is the constant coefficient of sliding friction, and $r$ is the radius of the shaft and of its hemispherical end.

**Force Required to Tighten up Bolts.** — Let

$P$ be the force exerted on the spanner,
$l$ the lever arm of the force $P$,
$Q$ the tension produced in the bolt,
$r$ the mean radius of the threads,
$r_1$ the external radius of the bolt,
$r_2$ the external radius of the bearing surface of the nut,
$\alpha$ the pitch angle of the threads,
$\phi$ the angle of friction.

Then, during a rotation of $\theta$ radians after the nut fairly bears, we have

$\frac{Q}{2}(r\theta \tan\alpha)$ = work done in stretching the bolt $\left(\text{where } \frac{Q}{2} \text{ is the average force}\right)$,

$\frac{Q}{2}\mu \left\{\frac{2}{3}\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}\right\}\theta$ = work done in overcoming pivot friction,

$\frac{Q}{2}\theta r\{\tan(\alpha+\phi)-\tan\alpha\}$ = work done in overcoming screw friction,

$\frac{Pl\theta}{2}$ = work done at the spanner.

Hence, by the principle of work, we have

$$\frac{Pl\theta}{2}=\frac{Q\theta}{2}\left\{r\tan\alpha+\frac{2\mu}{3}\left(\frac{r_2^3-r_1^3}{r_2^2-r_1^2}\right)+r\tan(\alpha+\phi)-r\tan\alpha\right\}$$

from which $P$ may be found.

EXERCISE 188. The dimensions of standard bolts are:

$d$ = the diameter of the bolt,
$.8\,d$ = the diameter at the root of the threads,
$.16\,d$ = the pitch of the threads,
$1.4\,d$ = the external diameter of the bearing surface of the nut.

The spanners used in tightening such bolts have an effective length of about $15\,d$. The angle of the thread in the U. S. Standard bolt is 60°.

(a) If $\mu$ is assumed as 0.15, find the tension produced in the bolt by a force of $P$ pounds on the spanner.

(b) If a workman is assumed to exert a force of 30 pounds upon spanners used in tightening bolts and the material of the bolt is assumed as capable of resisting a tension of 26,000 pounds per square inch with a factor of safety of 5, what is the least diameter of bolt that can safely be used?

EXERCISE 189. What force would loosen the nut in EX. 188 if it is tightened by a force of 30 pounds?

## SECTION XXI

### FRICTION OF BANDS (Ropes, Belts, etc.)

**Normal Force Due to Bands in Tension.** — As in Fig. 97, assume a band in contact with a fixed cylinder over an arc $\alpha r$ and that this band is under a tension $T$. Then as no relative motion tends to occur no frictional forces are called into action and the reaction of the cylinder is everywhere normal to the band.

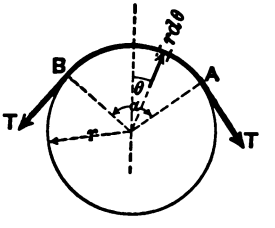


FIG. 97

Let $p$ = the normal force on the band per unit of length, then $pr\,d\theta$ is the normal force on any element, and as the forces acting on the band are in equilibrium the sum of the vertical components of all forces must be zero,

so that $$\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} (pr\,d\theta)\cos\theta - 2T\sin\frac{\alpha}{2} = 0,$$

or $$2pr\sin\frac{\alpha}{2} = 2T\sin\frac{\alpha}{2},$$

whence $$p = \frac{T}{r}.$$

Thus, **the normal force between band and cylinder per unit of length of the band equals the tension** in the band **divided by the radius of the cylinder.**

**The Effect of Friction.** — If the tension at the right-hand end of the band (Fig. 97) is increased the band does not necessarily slip towards the right. Friction is called into play along the arc $AB$.

The frictional force on an element at $A$ is greater than

the frictional force towards $B$, for the tension at $A$ is greater than the tension at $B$ and thus produces a greater normal pressure at $A$ than at $B$. Thus the friction along $AB$ gradually balances the excess of tension at $A$.

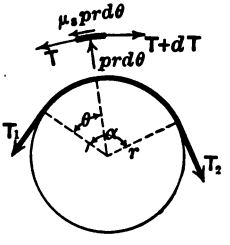

FIG. 98

In Fig. 98, consider $T_2 > T_1$ and let $\mu_s$ be the statical coefficient of friction. Then for impending motion the excess of tension on any element of the band must be just balanced by the limiting friction on this element. This limiting friction equals $\mu_s (pr\, d\theta)$, where $p = \frac{T}{r}$ the normal force per unit of length.

By equating moments about the center of the cylinder we obtain

$$dT = \mu_s \left(\frac{T}{r}\right) r\, d\theta$$

or
$$\frac{dT}{T} = \mu_s\, d\theta,$$

whence $\log_e T = \mu_s \theta + c,$

and as $T = T_1$ when $\theta = 0$, $c = \log_e T_1,$

and for an arc of contact, $\alpha$, when $T = T_2$,

we have
$$\log_e \frac{T_2}{T_1} = \mu_s \alpha$$

or
$$\frac{T_2}{T_1} = \epsilon^{\mu_s \alpha},$$

where $\alpha$ is measured in radians and $\epsilon = 2.718 \ldots$, the base of the Naperian system of logarithms.

EXERCISE 190. $T_2$ is how many times greater than $T_1$ for impending motion in the case of a hemp rope wrapped

about a wooden drum ($\mu_s = 0.4$) when the rope makes 0.5, 1, 2, 3.5 turns about the drum?

EXERCISE 191. For leather belts on iron pulleys $\mu_s$ may be taken as 0.25. What is the value of the ratio of the tensions at the two ends of a belt covering one-half the circumference of the pulley if relative motion is about to occur?

EXERCISE 192. If the tension on the tight side of a belt is $T_2$ and on the slack side is $T_1$ show that the *initial tension* $T_0$ (before driving occurs) must not be less than

$$\frac{T_2 + T_1}{2} = \frac{T_2 - T_1}{2} \frac{e^{\mu_s \alpha} + 1}{e^{\mu_s \alpha} - 1},$$

where $T_2 - T_1$ is the effective driving force.

**Energy Transmitted by Belts. Neglecting the effect of the velocity of the belt**, the energy transmitted would be

$$(T_2 - T_1)\, v = T_2 (1 - \epsilon^{-\mu_s \alpha})\, v \text{ foot-pounds per second,}$$

if $T_2$ = tension on the tight side of the belt, in pounds;
$T_1$ = tension on the slack side of the belt, in pounds;
$v$ = velocity of the belt at its mid thickness, in feet per second,
$\alpha$ = the lesser angle of contact between belt and pulleys, in radians;
$\mu_s$ = the statical coefficient of friction.

EXERCISE 193. For single-ply leather belting $T_2$ may be taken as 80 pounds per inch of width. The thickness of such belting is about 0.22 inch. Show that if $b$ = width of the belt in inches, $v$ its velocity in feet per minute, $\mu_s = 0.4$, and $\alpha = \pi$ radians, the H.P. transmitted is $\frac{bv}{600}$, approximately.

EXERCISE 194. Show that under the conditions of Ex. 193 the number of square feet of belting passing any point per minute per H.P. is 50.

**The effect of velocity on the tension and hence upon the energy transmitted by a belt** is considerable and should usually be allowed for in such calculations. That portion of the tension in a belt which supplies the necessary centripetal force whenever a belt moves in a circular path, sometimes called the **centrifugal tension**, can readily be computed as follows:

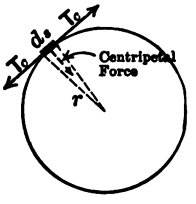

FIG. 99

Consider a differential length of the belt, $ds$, Fig. 99, moving in a circle of radius $r$ with a velocity $v$.

Then if $m$ is the mass of the differential element of the belt the centripetal force $= m\frac{v^2}{r}$.

Let $T_c$ be the tension which is just sufficient to furnish the necessary centripetal force, in pounds,

$w$ the weight of the belt in pounds per foot of length,

$v$ the velocity of the belt in **feet per second,**

$r$ the radius of the pulley in feet,

and $g$ the acceleration of gravity (32.2 feet per second per second).

Then the centripetal force per element of belt

$$= m\frac{v^2}{r} = \frac{w\,ds}{g}\left(\frac{v^2}{r}\right),$$

and as the centripetal (normal) force due to a tension $T_c$ per element of belt $= \frac{T_c}{r}ds$ (see page 135),

we have

$$\frac{T_c}{r}ds = \frac{w\,ds}{g}\frac{v^2}{r},$$

that $$T_c = \frac{w}{g} v^2.$$

If the tension in a belt is actually $T$ pounds and $T_c$ unds of this tension is required to supply the centrip- al force which causes the change in direction of the locity of the belt, there remains only $T - T_c$ pounds tension available for the production of the normal essure which in turn gives rise to the frictional forces cessary for driving.

Thus, instead of the formula

$$T_2 = T_1 e^{\mu_0 \alpha},$$

hich is used whenever a rope is about to slip over a ationary cylinder or pulley, we must use

$$\left(T_2 - \frac{wv^2}{g}\right) = \left(T_1 - \frac{wv^2}{g}\right) e^{\mu_0 \alpha},$$

enever the pulley and belt rotate with an appreciable ocity with $T_2$ as the total tension on the tight side and as the total tension on the slack side of the belt.

The energy transmitted by a belt is thus more accu ely

$$\left(T_2 - \frac{wv^2}{g}\right)(1 - e^{-\mu_0 \alpha})\, v \text{ foot-pounds per second.}$$

EXERCISE 195. If $W$ is the weight of belting in pounds r cubic inch, express $T_c$ in terms of $W$.

EXERCISE 196. Show that $T_c = 0.01\, bt v^2$ for leather lts, where $b$ is the width and $t$ the thickness of the belt in hes. (Leather weighs 56 pounds per cubic foot.)

EXERCISE 197. If a leather belt travels at a speed of 00 feet per minute when it ceases to drive the driven pulley, at is the tension in the belt at this time?

EXERCISE 198. Show that A. F. Nagle's Formula (Trans. A.S.M.E., Vol. II, 1881, page 91, or Kent, 8th Edition, page 1117) is equivalent to the formula above deduced.

EXERCISE 199. Show that the energy transmitted by a belt becomes a maximum for a velocity

$$v = \sqrt{\frac{T_2 g}{3w}},$$

and that the maximum energy transmitted is

$$\frac{2}{3} T_2 (1 - \epsilon^{-\mu_s \alpha}) \sqrt{\frac{T_2 g}{3w}} \text{ foot-pounds per second.}$$

EXERCISE 200. A leather belt which can stand a working tension of 80 pounds per inch of width is 0.22 inch thick and weighs 0.035 pound per cubic inch. Assuming $\mu_s = 0.4$, and the arc of contact 180°, find the maximum H.P. which the belt can transmit per inch of width and the velocity at which it must be run to transmit this power.

EXERCISE 201. What power would the belt transmit at the velocity computed in Ex. 200 if the centripetal force could be neglected?

**Belt Creep.** — Both the simple formula for the power transmitted by belts and the more exact one which takes into account the action of centripetal force were deduced under the tacit assumption that belts are inextensible. As all belts stretch under tension these formulas must be regarded as approximations which experiments show to be sufficiently accurate for most purposes.

Calculations involving the elasticity of materials (discussed in *Mechanics of Materials*) lie without the scope of this book, but a brief general description of the effect of elasticity upon the transmission of energy by belts may not be out of place.

It is evident that belts transmitting energy never do

so under the limiting conditions of motion assumed above. Either the limiting friction is not called into action or, as it will be more convenient for our analysis, it may be conceived that the limiting friction acts over only a portion of the arc of contact between pulley and belt. Over the remaining portions of the arc the friction may then be considered negligible. In Fig. 100 let the arc over which the limiting friction acts be $BC$. Then if $T_2 > T_1$ and motion is about to occur in the direction indicated by the arrow in that figure, the pulley represents the driven pulley, we may put

$$T_2 = T_1 e^{\mu\alpha'} \quad \text{or} \quad T_1 = T_2 e^{-\mu\alpha'}$$

where $\alpha'$ is the angle subtended by the arc $BC$ and $\alpha$ the angle subtended by the arc $AD$. The increase in tension from $T_1$ to $T_2$ occurs in the arc $BC$ while the tension throughout arc $AB$ is $T_1$ and throughout arc $CD$ $T_2$.

Fig. 100

Fig. 101

If now the pulley moves, no change occurs until $C$ reaches $D$. Any further motion must cause an increase in tension at $B$, Fig. 101, for the change from $T_1$ to $T_2$ cannot occur over an arc subtending an angle less than $\alpha'$. Thus the element of the belt arriving at $B$ must stretch under the increasing tension, for on leaving the pulley at $D$ its length corresponds to a tension $T_2$. Hence a greater length of belt leaves the driven pulley at $D$ than is taken on at $A$ during the same time.

Under these conditions the belt is said to **creep** from $B$ to $D$ in the direction of its motion. The creep must be carefully distinguished from ordinary slip which occurs when the difference in tensions, $T_2 - T_1$, is so great that the limiting friction over the whole arc of contact can no longer prevent the motion of the belt as a whole relatively to the pulley.

The slipping of belts is preventable by proper design, but the creeping of belts must occur.

It can be shown by a similar argument that the belt must creep in a direction opposed to its motion over a portion of the surface of the driving pulley.

Moreover, the driven pulley has an angular velocity corresponding to the linear velocity of that portion of the belt whose tension is $T_1$, while the driving pulley has a velocity corresponding to that portion of the belt whose tension is $T_2$.

All this leads to the conclusion that the driven pulley runs slower than calculations neglecting the creep would show, and as the effective tension $(T_2 - T_1)$ at driven and driving pulley is the same the efficiency of belt transmission must be less than 100 %. Experiments tend to show that the loss due to creep in properly designed belts should not be more than 1 or 2 %.

An interesting experiment illustrating creep may readily be performed by means of an india-rubber (highly elastic) belt passing over equal pulleys. Here the highly exaggerated creep is rendered strikingly obvious.

For further information on this and other problems relating to the transmission of energy by belts, especially at high speeds, the reader is referred to C. Bach, *Die Maschinen-elemente, Zehnte Auflage*, a paper by Wm.

W. Bird,* and especially to the work of C. O. Gehrckens† and of Kammerer‡ of Charlottenburg. A synopsis of some of the above references will be found in a paper entitled *A New Theory of Belt Driving* by Silby Haar.§

## SECTION XXII

### JOURNAL FRICTION

In order to confine the motion of a shaft to a rotation about its axis, the shaft is supported by **bearings.** That portion of the shaft which is in contact with the bearing is called the **journal.** Bearings and journals are usually cylindrical surfaces, the diameter of the journal being slightly smaller than that of the corresponding bearing. When the shaft rotates, sliding friction occurs at the surfaces of contact between journal and bearing.

Theoretical calculations involving the relation between the coefficient of sliding friction and the moment of the frictional resistances at journals necessarily depend upon the assumptions made in regard to the distribution of the pressure between the journal and its bearing.‖ Various results have been obtained under various assumptions much in the same manner as in the case of the friction of pivots. Attempts have also been made to investigate theoretically the difference between new and worn bear-

* Bird: Trans. A. S. M. E. **26,** 584. 1905.

† Gehrckens: Z. d. V. d. I. **37,** 15. 1893.
**52,** 1443, 1820. 1908.

‡ Kammerer: Z. d. V. d. I. **51²,** 1085. 1907.
**52,** 1444. 1908.
**53,** 661. 1909.

§ Haar: Journal A. S. M. E., Dec., 1910, p. 1997.

‖ C. Bach: "Die Maschinen-elemente, Zehnte Auflage," p. 494 et. seq., 1908.

ings. When these results are checked by experiment they are found unsatisfactory, so that it is usual to discard the theoretical formulas and base all calculations concerning journal friction directly upon experimental investigations.

We shall discuss as an illustration of the theoretical formulas only the simplest case, i.e., the case of

**The Worn Bearing.** — Let Fig. 102 (a) represent a journal at rest in a bearing whose wear has been much exaggerated. If the resultant force, $R$, upon the journal is vertical the reaction of the bearing will be vertical and the journal will occupy the lowest possible position in the bearing, as shown.

Assume now that the shaft rotates in the direction indicated by the arrow, the resultant of all forces upon it still remaining vertical. It does not now remain in position and rotate about its axis but rolls upon the surface of the bearing until the angle indicated by $\phi_s$ in Fig. 102 (b) becomes equal to the statical angle of friction. Then slipping occurs and the journal descends to a somewhat lower position (for $\phi_k < \phi_s$) and retains this position as it continues to rotate about its axis in this new position.

In Fig. 103 let $P$ represent the driving force and $Q$ the useful resistance applied to pulleys (not shown) fastened to the shaft whose journal friction is to be considered. Let $O$ be the point of contact between the journal and the bearing. Neglecting the weight of shaft and pulleys, equilibrium can only exist if the reaction of the bearing, $R$, passes thru the intersection of the lines of action of $P$ and $Q$ and of course thru $O$.

As the resultant force at the surface where sliding

occurs always makes an angle $\phi$ with the normal to these surfaces $R$ may be resolved into $R \cos \phi$, the normal force, and $R \sin \phi$, the frictional force acting upon the journal.

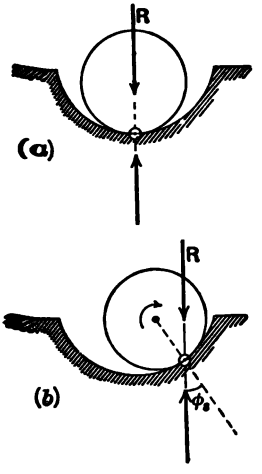


FIG. 102

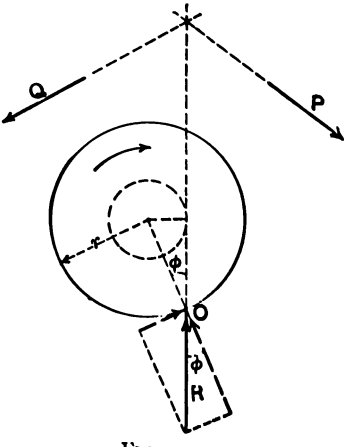


FIG. 103

The resisting moment due to friction is thus $(R \sin \phi)\, r$, where $r$ is the radius of the journal and $R$ is the resultant pressure upon the bearing.

EXERCISE 202. Here the frictional force equals $R \sin \phi$. Does this really differ from our previous expression for sliding friction?

For convenience in the **graphical solution** of problems involving journal friction the **friction circle** is used. This circle is drawn tangent to the line of action of $R$ in Fig. 103 and has the center of the journal as its center. The figure shows its radius to be $r \sin \phi$. As the point $O$

is in general undetermined the friction circle is always drawn concentric with the journal and of radius $r \sin \phi$ (or its equivalent, see page 148) and it then assists in locating the line of action of the force transmitted by the bearing to the journal.

Note carefully that the line of action of this force must be tangent to the friction circle upon that side which will cause the reaction to resist the motion.

EXERCISE 203. A shaft makes $n$ r.p.m.; if one of its journals is $d$ inches in diameter and transmits a pressure of $R$ pounds to its bearing, find the H.P. lost in friction at this bearing.

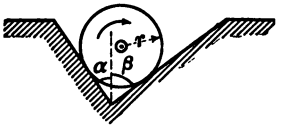

FIG. 104

EXERCISE 204. Find the work lost per minute in friction at a journal revolving at $n$ r.p.m. in a v-bearing, Fig. 104, if $R$ pounds is the total vertical pressure on the journal whose radius is $r$ inches, the angle of friction being $\phi$.

As already stated the coefficient of sliding friction, or its corresponding angle, is not used in computations involving journal friction as would be indicated by the above solutions.

This is due in part to the assumptions which must necessarily be made in arriving at these theoretical formulas, also to the fact that journals are usually subjected to much more effective lubrication than surfaces at which ordinary sliding friction occurs. This changes the conditions under which the coefficients are determined to such an extent that new **coefficients of journal friction** must be found experimentally. These will be found in engineering hand-books (see page 104).

Morin made some of the first experiments on journal friction. His results were obtained under conditions of velocity, pressure and lubrication hardly consistent with modern practice. Morin found that the friction was independent of the velocity and proportional to the load, the coefficient $\mu = \frac{F}{P}$, depending only upon the nature of the surfaces and upon their lubrication.

G. A. Hirn* published the results of his friction of lubricated journals in 1854, showing that the friction decreased with increasing temperature. Thurston† also made extensive experiments on the lubrication of and the frictional resistances in journals under various conditions of velocity, pressure and temperature. Perhaps the most extensive experiments were carried out under the committee on friction of the Institution of Mechanical Engineers by Beauchamp Tower.‡ Goodman§ reviewed the work of Thurston and Tower in Trans. Inst. C.E., 1886.

For references to recent theoretical and experimental investigations on journal friction the reader is referred to the *Engineering Index*.

Experiments have shown that the **coefficient of journal friction** $\mu$, depends upon the pressure, the velocity of sliding, the lubricant and method of lubrication, the temperature, and the design of the bearing. It should

* Hirn: Société Industrielle de Mulhouse, 1855.

† Thurston: "Friction and Lost Work," 1885.

‡ Tower: Proc. Inst. Mech. Eng. —, 632, 1883.
—, 58, 1883.
—, 173, 1888.

§ Goodman: Eng'g News, 19, 241, 304, 388, 1888.
Browne: Engineer, 58, 57, 1884.

be remembered that in thoroly lubricated journals the friction is really not friction between solids, but fluid friction, for the lubricant forms a film between the bearing and journal.

In view of the foregoing it is found advisable to **use $\mu_f r$ instead of $r \sin\phi$ as the radius of the friction circle,** where $r$ is the radius of the journal and $\mu_f$ is the coefficient of journal friction to be determined as such by experiment.

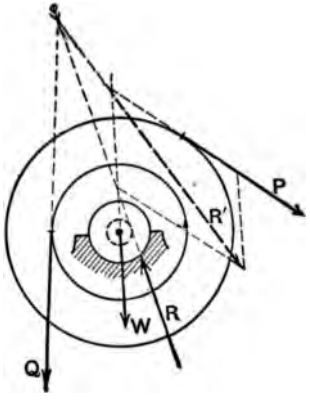


FIG. 105

As an **application of the friction circle** consider the wheel and axle shown in Fig. 105. Let $P$ be the driving force, $Q$ the useful resistance, and $W$ the weight of the wheel, axle, and shaft. The diameters of both the journal and its friction circle, whose radius is $\mu_f r$, have been purposely exaggerated for clearness.

$P$ and $W$ are known; $Q$ and $R$, the pressure on the bearing, are required.

Find first the resultant, $R'$, of $P$ and $W$. Then the line of action of the force on the journal, $R$, must pass thru

the intersection of the lines of action of $R'$ and $Q$ and it must also be tangent to the friction circle. Moreover, as the force $R$ transmitted by the bearing to the journal is upward and as the motion considered is clockwise, the point of tangency must lie on the right side of the friction circle so that $R$ may oppose the motion.

A triangle of forces drawn with $R'$ as base and whose sides are parallel to $R'$, $R$, and $Q$ will complete the solution.

EXERCISE 205. What changes in Fig. 105 would be necessary

(a) for backward motion, i.e., $Q$ overcoming $P$,

(b) for frictionless motion?

EXERCISE 206. Find the efficiency of a wheel and axle (radii 4′ and 2′), the radius of the journal being 6 inches and $\mu_j = 0.2$; the useful resistance at the axle is vertical and equal to 100 pounds, while the driving force is horizontal for (a) forward motion, (b) backward motion.

EXERCISE 207. Find the vertical force required to (a) lower, (b) raise a load of 300 pounds by means of a fixed pulley 4 feet in diameter having a journal 6 inches in diameter with $\mu_j = 0.1$. Solve both graphically and analytically.

From the solutions of the above example and exercises note that whenever the line of action of the resultant pressure, $R$, transmitted from the bearings to the journals of a shaft forms a secant to the friction circle no motion is possible. If the applied forces are so changed as to cause the line of action of the force $R$ to become tangent to the friction circle motion is about to occur, or continues with constant velocity. If the line of action of $R$ neither intersects nor touches the friction circle accelerated motion occurs.

# CHAPTER VII

# FURTHER APPLICATIONS TO MACHINES

## SECTION XXIII

### MEMBERS OF MACHINES ACTED ON BY TWO FORCES

IN kinematics of machines those parts or members between which there exists **surface contact** are usually called **lower pairs.** There are only three ways in which lower pairing can occur, i.e., thru the action of

(a) **sliding pairs,** such as the piston and cylinder or the crosshead and its guides in the steam or gas engine;

(b) **turning pairs,** such as a journal and its bearing or a loose pulley on a shaft;

(c) **screw pairs,** such as the common screw and its nut.

Either part of any one of the above lower pairs is called **a kinematic element** and any part or member of a machine carrying two such elements is called a **link.**

The main mechanism of the ordinary steam engine (Fig. 106) contains four links, each of which carries two elements.

The link *a* carries guides at 1 and a bearing at 2; link *b* carries a journal at 2 and another journal at 3; link *c* carries a bearing at 3 and another bearing at 4; and link *d* (i.e., the block) carries one element of the sliding pair at 1 and a journal as an element of a turning pair at 4.

The link *d* (Fig. 106) is usually expanded so as to comprise the piston, piston-rod, and crosshead; this in no way affects the kinematics of the machine, while it does alter the forces acting upon this link.

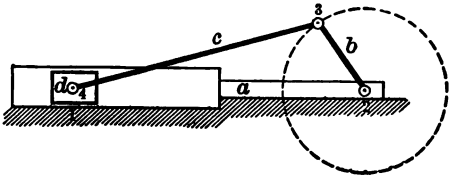


FIG. 106

The forces transmitted by sliding pairs have been considered on page 111, and on page 143 the forces acting on turning pairs were discussed. It remains to apply these results to some problems involving several of these pairs.

Consider first the action of a **link carrying one member of each of two turning pairs** and acted on by no forces, except such as are **transmitted** by these turning pairs.

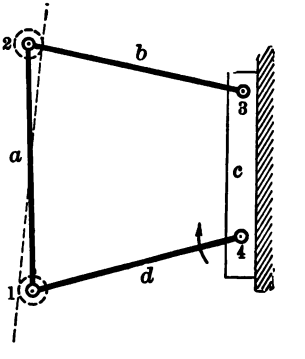


FIG. 107

To fix our ideas, consider link *a* of the machine illustrated in Fig. 107 as the link carrying one element (say the bearings) of each of the turning pairs 1 and 2. Here link *c* is the **fixed link** or **frame** of the machine.

Also assume the link *d* to drive in the direction indicated by the arrow.

From the discussion on page 145 we know that the

force transmitted from link *d* to link *a* thru the turning pair 1 must have its line of action tangent to the friction circle of the turning pair at 1. This friction circle (dotted in Fig. 107) has been much enlarged for the sake of clearness; its radius should equal the product of the radius of the journal at 1 and the coefficient of journal friction $\mu_j$ as found by experiment.

We must now determine upon which side of this friction circle the point of contact between it and the line of action of the transmitted force lies. This depends upon two considerations: first, **the direction of the force transmitted**; and second, **the direction of rotation in the turning pair.** Great care must always be taken to consider the motion of the link, upon which the force is supposed to act, relative to the motion of the other link.

In our particular illustration when the force transmitted from link *d* **to link** *a* is used, then the **motion of link** *a* **relative to** link *d* must be used simultaneously. Here the force transmitted **to link** *a* is upward and link *a* moves in a counter-clockwise direction relative to link *d* when this link drives as shown by the arrow. Therefore, this **upward force** upon link *a* must act to the left of the center of the turning pair 1 in order to **oppose** the **counter-clockwise motion** of the link *a*.

We might also have argued thus: The force acting at 1 upon **link** *d* is downward. The **motion of link** *d* relative to link *a* is clockwise. Therefore, in order to **oppose** the **clockwise motion** of link *d* the **downward force** acting upon it must act to the left of the center of the turning pair at 1.

Next draw the friction circle for the turning pair at 2. Then as link *a* acts on link *b* thru 2 in an upward direc-

tion and link *b* moves clockwise relatively to link *a*, the force must act to the right of center 2 in order to oppose the motion. Or, again, link *b* presses downward upon link *a* and link *a* moves counter-clockwise relatively to link *b*; therefore, the point of contact must lie to the right of center 2.

As no force other than the reactions at 1 and 2 act upon the link *a* (its weight neglected) these forces must, for equilibrium, oppose each other along the same line of action, and this line of action must be the common internal tangent of the friction circles, tangent to circle 1 on the left and tangent to circle 2 on the right, as shown.

The line of action of the force transmitted by the link is called the **friction axis of the link.**

EXERCISE 208. Show clearly the line of action and direction of the forces acting upon

(a) link *a* (Fig. 107) at 1 and 2,
(b) link *d* at 1,
(c) link *b* at 2.

EXERCISE 209. Sketch the friction axis of the link *a* in each of the machines shown in Fig. 108. The arrow in each case indicates the driving link and its direction of motion.

EXERCISE 210. Sketch the friction axis of the link *c* in the chain illustrated in Fig. 106 (which represents the mechanism of the steam engine); assume the engine to run clockwise and make four sketches showing the friction axis when the link *b* lies in each of the four quadrants.

Return now to the problem illustrated in Fig. 107 and let *P* (Fig. 109) be the driving force, which is wholly known, and *Q* the useful resistance, unknown in magnitude. Assuming all links as weightless, find all forces

acting upon the machine. As already explained (page 153) the friction-axis is the line of action of the force $L$ transmitted by the link $a$.

Consider the forces acting upon link $d$. Here $P$ is wholly known, $L$ is known as regards its line of action

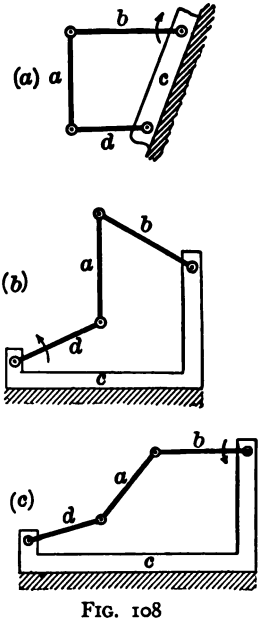


FIG. 108

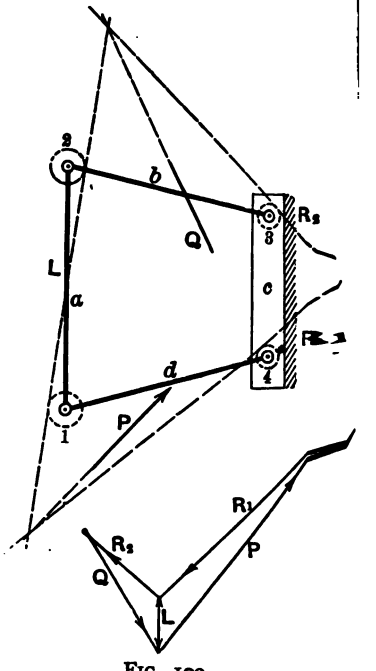


FIG. 109

and direction (downward upon link $d$), and $R_1$, the reaction at 4, is wholly unknown.

As only three forces act upon link $d$ the intersection of $L$ and $P$ gives a point on the line of action of $R_1$. Also $R_1$ must be tangent to the friction circle at 4. The general direction of the resultant of $L$ and $P$ being towards the right, $R_1$, equal and opposite to this resultant,

must act towards the left. To oppose the motion of the link $d$ it must be tangent to the friction circle 4 on the upper side. It also passes thru the intersection of $P$ and $L$. The line of action of $R_1$ is thus determined.

With $P$ as a base the triangle of forces $P$. $L$. $R_1$ can now be constructed, determining $L$ and $R_1$.

Passing to the link $b$, we have the direction of the lines of action of $L$ and $Q$; their intersection is a point on $R_2$. $L$ acts upward upon link $b$ and $Q$ acts downward opposing the force $P$ and the motion of link $b$ : their resultant acts towards the right; thus $R_2$ for equilibrium must act towards the left. To oppose the motion of link $b$. $R_2$ must thus be tangent to the friction circle at 3 on its upper side.

Upon $L$ as base the triangle of forces $L$. $Q$. $R_2$ can now be constructed. This triangle determines the magnitudes of $Q$ and $R_2$.

EXERCISE 211. Make separate sketches showing links $a$, $b$, and $d$ (Fig. 109), as free bodies. Carefully indicate the directions of all forces.

EXERCISE 212. Assume the magnitude, line of action, and direction of a useful resistance, and the line of action of a driving force in Fig. 108 (b) and (c) and find all other forces.

As another important link occurring in machines, consider a link joining a sliding pair and a turning pair. A link of this nature carries one member of the turning pair (the bearing) and one of the two sliding surfaces forming the sliding pair. This link should also be conceived as acted upon only by the forces transmitted by the pairs joined by it; thus it is acted on by two forces only.

An illustration of this kind of link is furnished in the

mechanism for transforming rectilinear motion into circular motion by means of a slotted crosshead.

In Fig. 110 the block sliding in the slotted crosshead is the link under consideration. This link carries the bearing in which the crank pin turns.

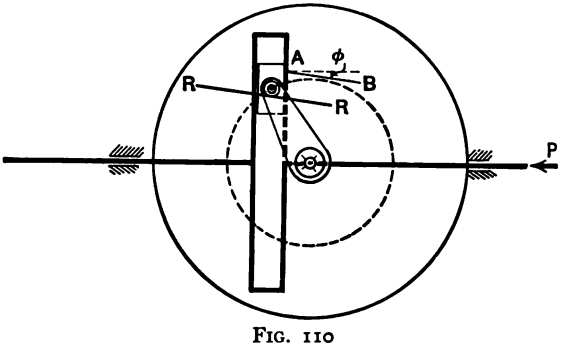


FIG. 110

If the crosshead drives towards the left as indicated by the arrow $P$, then the block is pushed towards the left and contact occurs on its right side. The force transmitted **to the block** by the crosshead will be towards the left. The motion **of the block** relative to the crosshead will be downward. Therefore, the force acting on the block will be inclined to the normal of the sliding surfaces at an angle $\phi$ as shown by the line $AB$ in Fig. 110. The line of action of this force is not yet determined; only its direction is known; it must be parallel to $AB$.

The block is acted on by another force, namely, the force transmitted by the crank-pin journal to its bearing in the block. As the crank resists the counter-clockwise rotation impressed upon it the force transmitted **to the block** will be towards the right. The motion **of the block** relative to the crank is one of clockwise rotation.

To hinder this rotation the force toward the right must be tangent to the lower side of the friction circle. The direction of this force is unknown, but its line of action must be tangent to the friction circle of the lower side.

As only two forces act upon the block (its weight is neglected) these forces must have the same line of action. Thus the only line of action satisfying the requirements of both sliding and journal friction is the line *RR*, Fig. 110.

EXERCISE 213. Show the direction of the force transmitted by the block

(a) to the slotted crosshead

(b) to the crank.

Do these forces oppose motion?

(c) Show the forces acting upon the block.

(d) What would be the direction and position of the line *RR*, Fig. 110, if all surfaces were smooth?

EXERCISE 214. Find the line *RR*, Fig. 110, if the crank drives the crosshead

(a) towards the left,

(b) towards the right.

EXERCISE 215. Assume a vertical resisting force $Q$ tangent at the left to the largest circle in Fig. 110 (representing a pulley) and find $Q$ for a given driving force $P$. Consider the crank, pulley, and shaft as of one piece and neglect all weights.

EXERCISE 216. Solve Ex. 215 when the block lies in the second quadrant of the return stroke.

EXERCISE 217. If, in Ex. 215, $Q$ drives, find $P$ for a given $P$.

EXERCISE 218. In Ex. 215 indicate the direction and magnitude of the forces acting upon the bearings supporting the shaft and upon the guides supporting the slotted crosshead and its rod.

## SECTION XXIV

### GENERAL PROCEDURE FOR ANALYZING THE FORCES ACTING UPON ANY MACHINE

I. Consider the machine frictionless and find all forces acting upon the various members.

(a) Find all links acted upon by two forces only and determine their line of action.

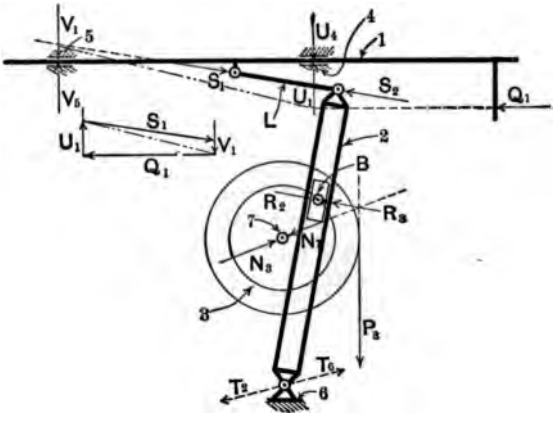


FIG. 111

(b) Start with that member of the machine upon which the given force acts and determine all other forces acting upon it by means of the conditions of equilibrium.

(c) Proceed from member to member until all members have been considered.

(d) Determine where contact occurs and the direction of the forces on the various members at these points of contact.

II. Introduce the modifications due to friction, paying careful attention to the information developed during

the analysis under I, and find the magnitudes of the new forces which now replace the forces found in I.

As an example find the force $P_3$ necessary to drive the mechanism shown in Fig. 111 against a given resistance $Q_1$.

This mechanism consists of the following parts: the sliding rod (1), the link $(L)$, the slotted crank (2), the sliding block $(B)$, the crank disk, pulley and shaft (3), the guides (4) and (5) supporting the sliding rod, the bearing (7) supporting the shaft, and the bearing (6) supporting the slotted crank.

The mechanism is first assumed frictionless.

The links acted on by only two forces are $(L)$ and $(B)$, weight neglected.

The line of action of the force transmitted by link $(L)$ coincides with its center line; i.e., along $S_1S_2$. The line of action of the force transmitted by link $(B)$ must pass thru the center of its bearing and be normal to its sliding surfaces, along the line $R_2R_3$.

As the force $Q_1$ is given, start with the member (1) upon which it acts and determine all forces acting upon this member.

Let $S$ be the force transmitted by $(L)$; then $S_1$ indicates the line of action and direction of the force transmitted by $(L)$ to (1); also let $S_2$ be the force transmitted by $(L)$ to (2). Here $S_1$ equals $S_2$ and $(L)$ is in tension to the extent of $S$ $(= S_1 = S_2)$ pounds. Link $(L)$ need be considered no further.

**Notation.** — Note that the capital letters denoting forces acting on (1) will carry a subscript 1, the capital letters denoting forces acting on (2) will carry a subscript 2, etc. As action and reaction are equal the same letters

will be used in denoting such pairs of forces. Thus, all forces designated by the same letter will be equal in magnitude irrespective of their subscripts.

Not only do $Q_1$ and $S_1$ act on the member (1) but other forces are transmitted to it by the guides (4) and (5). The lines of action of these forces are vertical; their directions and magnitudes are unknown. The magnitude of $S_1$ is also unknown. Thus (1) is in equilibrium under the action of four forces. The standard construction for four forces (see page 50) shows that $U_1$ acts upward and $V_1$ downward. Therefore contact occurs between (1) and the **lower guide at (4)** and between the rod and the **upper guide at (5)**.

EXERCISE 219. If $Q_1$ in Fig. 111 is changed so as to act horizontally

(a) along the rod (1),

(b) above the rod (1),

find the surfaces of contact at the guides (4) and (5).

EXERCISE 220. If friction occurs at the guides (4) and (5), Fig. 111, indicate the directions of the forces transmitted to the rod.

EXERCISE 221. Same as Ex. 220 under the conditions of Ex. 219.

Pass now to the member (2). The forces acting upon it are due to the contact of (2) with (*L*), (*B*), and (6). (*L*) acts upon (2) with the force $S_2$, the magnitude of which ($= S_1$) has already been determined. (*B*) acts on (2) with a force whose line of action is along the line $R_2R_3$ but this force is otherwise as yet unknown. Finally the third force acting on (2) is $T_2$ wholly unknown, except that its line of action must pass thru the center of the bearing at (6).

EXERCISE 222. Assume the inclination of $S_2$ to be such that the lines of action of $S_2$ and $R_2$

(a) meet to the right of (2),

(b) meet to the left of (2),

(c) are parallel,

and find $R_2$ and $T_2$ by means of freehand sketches.

The results of Ex. 222 show that $R_2$ is directed towards the right. Thus, as $(B)$ pushes (2) towards the right, contact must occur along the right side of $(B)$.

Note also from Ex. 222 that $T_2$ always has a horizontal component towards the left. Thus $T_2$ may in general be said to be directed towards the left.

The link $(B)$ need not be considered further; it is in compression to the extent of $R\ (= R_2 = R_3)$ pounds.

EXERCISE 223. If sliding friction occurs between $(B)$ and (2), Fig. 111, sketch the new line of action of $R$.

If journal friction occurs between $(B)$ and (3) sketch the new line of action of $R$.

Sketch the new line of action of $R$ for both sliding and journal friction.

EXERCISE 224. Referring to the solutions of Ex. 222 show how

(a) friction between (6) and (2),

(b) friction between (6) and (2),

in addition to all friction indicated in Ex. 223, will affect each of these solutions.

Next, member (3) is acted on by $R_3$ equal in magnitude to $R_2$ determined above, by $P_3$ unknown in magnitude, and by a force $N_3$ transmitted to it by the bearing (7) and passing thru the center of the bearing.

EXERCISE 225. Find $N_3$. Note that its direction is towards the right.

EXERCISE 226. Modify the solution of Ex. 225 so as to include friction between (7) and (3).

Having found the directions of all forces and the surfaces of contact involved when the machine is frictionless, proceed now to the systematic consideration of the effect of friction.

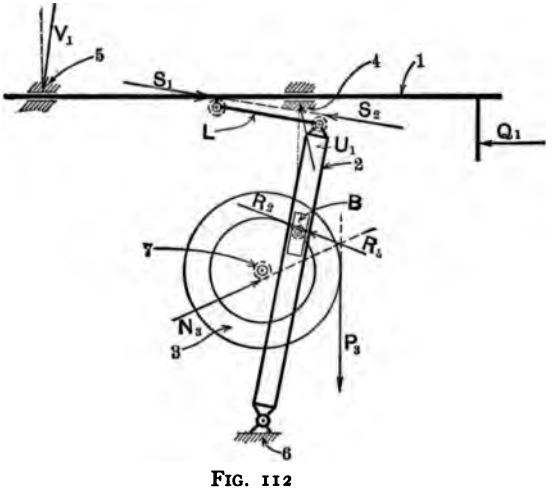


FIG. 112

In Fig. 112 the sketch of the mechanism is repeated. Here all dotted circles represent the friction circles of the turning pairs, all much exaggerated for the sake of clearness.

Again start with links ($L$) and ($B$). The forces $S_1$ and $S_2$ now act along the friction axis of the link. (See page 153.) The forces $R_2$ and $R_3$ act along a line determined by the method outlined on page 156.

In determining these lines of action great care must be taken to consider the forces acting on the link together

with the motion of the link. The forces $S_2$, $S_3$, $R_2$, and $R_3$ must not be used unless the motions of the members upon which these forces act relative to the link considered are also used.

For instance, $(B)$ pushes (2) towards the right (Fig. 111), but (2) pushes (B) towards the left, and as (B) moves downwards relatively to (2) the new line of action must be inclined below the normal on the right side at an angle $\phi$.

Again, $(B)$ pushes (3) towards the left, Fig. 111, but (3) pushes (B) towards the right. As $(B)$ rotates counter-clockwise relatively to (3) the line of action must be tangent along the upper side of the friction circle so as to hinder this motion.

Fig. 112 sufficiently explains the other modifications due to friction.

EXERCISE 227. Find all forces acting upon the members of the machine illustrated in Fig. 112, including all frictional resistances.

EXERCISE 228. Sketch all parts of the machine illustrated in Fig. 112, as well as all points of support (fixed points on the machine frame) and show the forces acting on each part.

EXERCISE 229. Consider the machine, Fig. 111, frictionless, with the slotted crank in the position shown, but during its return stroke. Find the resistance at $Q$ necessary to balance a given driving force at $P$.

EXERCISE 230. Same as Ex. 229, but including all frictional resistances.

EXERCISE 231. Find $Q$ for a given $P$ in Fig. 113, the machine to be considered frictionless.

EXERCISE 232. Same as Ex. 231, but all frictional resistance to be included.

In some cases the weights of certain members of a machine are of such magnitudes as compared to the forces transmitted that these weights must be taken into account.

For instance, the large circle in Fig. 113 may represent a fly-wheel to the periphery of which the force $Q$ is applied.

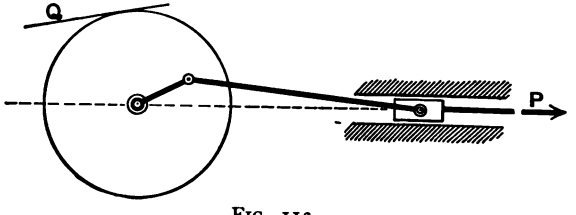


FIG. 113

To include this known force in the construction combine it with the force transmitted to the rotating parts by the connecting-rod. This resultant force may now be regarded as the only known force applied to the rotating parts of the machine and the usual method may be used to determine $Q$ and the bearing reaction.

EXERCISE 233. Modify the solution of Ex. 232 so as to include a given weight of fly-wheel.

EXERCISE 234. Assuming a weight for the fly-wheel, Fig. 113, and a force $P$, find $Q$ when the crank has made a half revolution from the position shown in the figure,

(a) assuming the machine to be frictionless,

(b) including all frictional resistances.

(c) What forces determined in (a) and (b) may be used to compute the efficiency of the machine for the phase of its cycle under consideration?

EXERCISE 235. In Fig. 114 assume the resistance $Q$ and find the force $P$ required to overcome this resistance, all *frictional* resistances to be included.

EXERCISE 236. Sketch each member of the machine, Fig. 114, and show all forces acting upon these members.

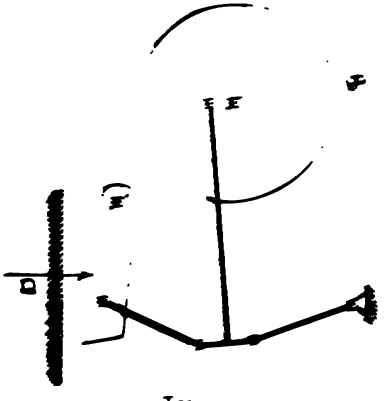


FIG. 114

EXERCISE 237. Sketch the frame of the machine, Fig. 114, and show all forces acting upon it.

EXERCISE 238. In Fig. 115 find $P$ for a given $Q$.

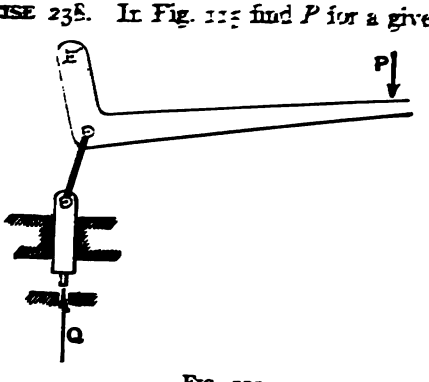


FIG. 115

In Fig. 116 is shown an oscillating steam cylinder driving a crank. Note that the steam acts both on the piston and on the cylinder head. When frictional resistances are considered, the piston-rod, piston, and

cylinder should be regarded as a link joining the trunnion $A$ (supporting the cylinder) and the crank pin $B$. The friction axis of this link can then be determined. Next consider the piston-rod and piston as a free body; note the reactions introduced by reason of the friction at the stuffing box $C$, and at the piston circumference.

EXERCISE 239. Assuming $P$, find $Q$, Fig. 116, friction neglected.

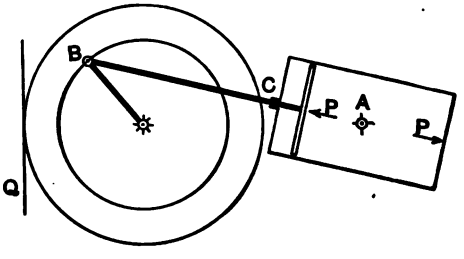


FIG. 116

EXERCISE 240. Solve Ex. 239, all frictional resistances included.

EXERCISE 241. Show the forces acting upon the frame of the machine illustrated in Fig. 116.

# CHAPTER VIII

# RESISTANCE TO ROLLING

## SECTION XXV

### INTRODUCTION

WHENEVER a cylinder rolls upon a plane or another cylindrical surface a certain resistance to the motion occurs.. Consider a cylinder resting upon two horizontal rails, Fig. 117, and supporting a perfectly flexible cord weighted with equal weights, $W$, as shown. If there existed no resistance to rolling the smallest additional mass added to either weight $W$ would cause accelerated motion of the cylinder.

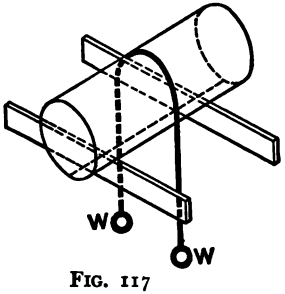


FIG. 117

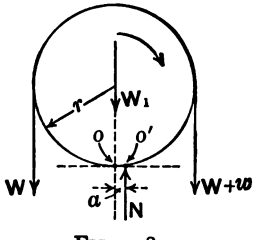


FIG. 118

EXERCISE 242. Compute the angular acceleration of the cylinder (mass, $m_1$; radius, $r$) shown in Fig. 117, due to masses $M$ and $M + m$ suspended at $W$ and $W$ respectively, resistance to rolling neglected.

Experiment shows that a slight increase in either weight $W$ is not sufficient to produce motion, or maintain

motion with constant velocity. This is explained by considering the reaction of the rails, $N$, displaced as shown in Fig. 118. The displacement, $a$, is called the **coefficient of resistance to rolling.** It is not a pure number without physical dimensions such as the coefficient of friction $\mu$; it is a length.

If the excess in weight, $w$, is just sufficient to maintain motion with constant velocity then by means of the conditions of equilibrium we obtain

$$a = \frac{w}{N} r = \frac{w}{2W + W_1 + w} r.$$

EXERCISE 243. Deduce the above equations.

From this equation $a$ may be computed. It has been found to equal

0.06 feet for wood on wood,
0.005 to 0.002 feet for metal on metal.

For wagon wheels

on soft soil $a = 0.065$,
on smooth hard roads $a = 0.02$.

Coulomb found that $a$ is independent of $r$. However, as $a$ must always remain a very small fraction of the circumference of the cylinder it is evident that $a$ must diminish as $r$ becomes smaller.

Later Dupuit found that $a$ is independent of the velocity but proportional to the square root of the diameter of the roller.

As the resistance to rolling is very small and as the experiments for its determination involve other frictional resistances or the stiffness of cordage, the accurate de-

termination of $a$ is very difficult and the laws governing this resistance are as yet imperfectly known.

To conceive of the nature of this resistance to rolling remember that, no matter how hard both roller and rails may be, the contact can never be a line contact. The roller and rails must be deformed and surface contact must occur. This deformation of continually new surfaces as rolling progresses must involve a waste of work, for no substance is perfectly elastic. Also the compression of the roller in a radial direction involves a lengthening in the circumferential direction. All portions of the roller are thus lengthened circumferentially when they come in contact with the rails; they contract again on leaving the rails. Similar changes, altho not necessarily of equal magnitude, occur in the rail. The net result causes the displacement of the wheel along the rail, per revolution, to be slightly shorter than the circumference of the wheel, and causes a continuous creeping of the wheel over the rail very like the creeping of belts over their pulleys when power is transmitted. (See page 140.)

These complicated actions occasion a loss of energy and may all be conceived as assisting in causing the displacement of the reaction between roller and rails as indicated in Fig. 118.

Returning to Fig. 118, assume two equal and oppositely directed forces each equal in magnitude to $N$ and parallel to $N$, applied at $O$; then the effect of the resistance to rolling can evidently be regarded as a couple whose moment is $Na$ (equal here to $wr$) and which opposes the motion. The total action of the rails upon the roller must now be represented by the normal reaction of the ground, $N$, passing thru $O$, and the couple $Na$.

As an application consider the motion of a cylinder placed upon an inclined plane, Fig. 119.

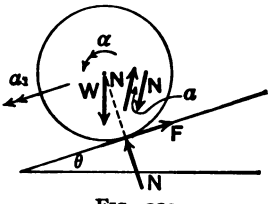

FIG. 119

The forces acting upon the cylinder are $W$, its weight; $N$, the normal reaction of the inclined plane; $F$, the frictional reaction of the plane; and, as resistance to rolling is to be included, a couple $Na$, which hinders the motion, must be added.

If $a_1$ is the acceleration of the mass center,
$\alpha$, the angular acceleration about the mass center,
$r$, the radius of the cylinder,
$m$, its mass,
$I$, the moment of inertia of the cylinder about its geometrical axis,
then the equations of motion are

$$Fr - Na = I\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$W \sin\theta - F = ma_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$W \cos\theta - N = m\,(0) = 0. \quad . \quad . \quad . \quad (3)$$

From these equations we have

$$N = W \cos\theta,$$

$$a_1 = g \sin\theta - \frac{F}{m},$$

$$\alpha = \frac{Fr - Na}{I}.$$

If the cylinder is to roll down the plane without slipping, the component acceleration along the plane of the element of the cylinder in contact with the plane must be zero, so that

$$a_1 - \alpha r = 0.$$

ERCISE 244. Show that under these conditions the on between the cylinder and the plane must be

$$F = \frac{1}{3} W \cos\theta \left(\tan\theta + 2\frac{a}{r}\right).$$

ember that the $I$ of the cylinder about its axis equals

ERCISE 245. Is $F$ always equal to $\mu_s W \cos\theta$ or os $\theta$?

w that for pure rolling, the slope of the plane is

$$\tan\theta \leqq 3\mu_s - 2\frac{a}{r}.$$

ler these conditions how does the value of $\theta$ affect the on $F$?

ERCISE 246. Show that under the conditions imposed 245

$$a_1 \leqq \frac{2}{3} g \left(\sin\theta - \frac{a}{r}\cos\theta\right),$$

$$\alpha = \frac{a_1}{r}.$$

ERCISE 247. Show that a cylinder dragged along a ntal plane by a horizontal force acting on its axis will s well as roll when

$$\text{the tractive force} > W\left(3\mu_s - 2\frac{a}{r}\right),$$

en

$$\mu_s < \frac{2}{3}\frac{a}{r}.$$

**the graphical solution of problems** of equilibrium ring resistance to rolling it is more convenient to ne the normal reaction $N$, Fig. 119, with the couple pposing the rotation into a single force equal in tude and parallel to $N$ but displaced a distance $a$

in such a direction as to hinder the rotation of the body upon which it acts.

EXERCISE 248. Write the equations of motion for the cylinder shown in Fig. 119 after replacing $N$ and $Na$ by a single force.

EXERCISE 249. Show graphically that if the cylinder, Fig. 119, rolls down the plane with constant velocity and the limiting sliding friction is called in action, then

$$\theta = \phi_s$$

and

$$\mu_s = \frac{a}{r}.$$

## SECTION XXVI

### APPLICATIONS

As an **example of the graphical treatment** of problems involving resistance to rolling, consider a wagon wheel rolling along horizontal ground. In Fig. 120 the line of action of the force exerted upon the wheel by the ground must pass thru the point $O'$ at a distance $a$ **in advance** of the point $O$, this point being on the line of action of the force when rolling resistances are neglected. The coefficient of rolling resistance, $a$, takes care of all resistances due to rolling, even those occasioned by the formation of a rut as illustrated in Fig. 120.

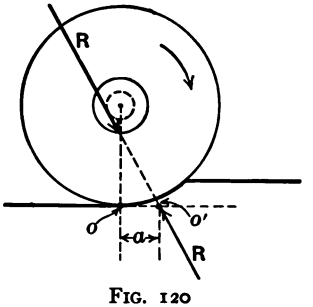


FIG. 120

Neglecting the weight of the wheel the only other force acting upon the wheel is the force transmitted to it by the journal on the axle of the wagon. The effect of

friction on this turning pair is taken into account by the friction circle. The line of action of the two forces which alone act upon the wheel must pass thru $O'$ and be tangent to the friction circle of the turning pair. To determine upon which side of the friction circle this line of action must be tangent note that the force on the wheel at the axle is downward and that the wheel revolves clockwise; thus to hinder this motion the down-

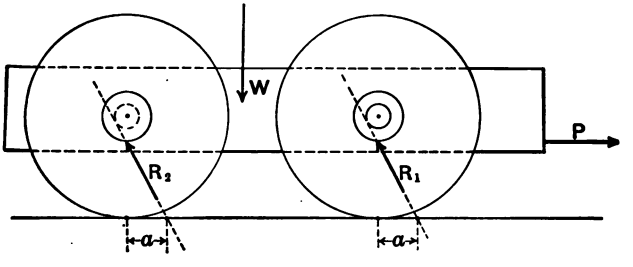


FIG. 121

ward force must be tangent to the friction circle at its left side. The line of action of $R$ is thus determined.

Let us now find the horizontal force required to keep a wagon, having four wheels of equal size, moving with constant velocity. If we assume the radii of the journals equal (Fig. 121) and also the same $a$ and $\mu_j$ for both pairs of wheels, then $R_1$ and $R_2$, the forces transmitted by the wheel pairs to the body of the wagon, are parallel. The wheels are here considered as links of the mechanism.

EXERCISE 250. Assume a load $W$ and its center of gravity, and find $P$ graphically.

EXERCISE 251. If $r$ is the radius of the journals and $r_1$ the radius of the wheels, show by means of Ex. 250 that

$$P = W \frac{a + r\mu_j}{r_1},$$

approximately. (Use similar triangles and remember that $a$ and $r\mu_j$ are very small as compared to $r_1$. These are much exaggerated in Fig. 121.)

EXERCISE 252. Show that

(a) the larger the diameter of the wheel, and

(b) the smaller the diameter of the journal

the less will be the tractive force.

EXERCISE 253. A wagon whose rear wheels are 4 feet in diameter and whose front wheels are 3 feet in diameter carries a load of 2 tons. The center of gravity of the load is 2 feet forward of the rear axle and the axles are 6.5 feet apart. The axle journals are 2 inches in diameter. Compute the tractive force for constant velocity on a level road if $a = 0.02$ and $\mu_j = 0.05$.

EXERCISE 254. Solve Ex. 253 graphically.

EXERCISE 255. Two rollers are used in moving a load $W$ supported on skids; show that

$$P = W\frac{a + a'}{2r},$$

where $P$ is the horizontal tractive force, $r$ the radius of the rollers, and $a$ and $a'$ are the coefficients of rolling resistance between the ground and the rollers and between the skids and the rollers respectively.

EXERCISE 256. In Ex. 255 does $P$ change with the changing positions of the rollers as the skids advance?

Does the height of $P$ above the ground affect the magnitude of $P$? Does it affect the problem in any way?

EXERCISE 257. Find the horizontal components of the reactions between ground and rollers in Ex. 251. This is the real resistance to rolling. Is this force in any way similar to the adhesion between the rails and drivers of a locomotive (see page 39)? Does the tractive force of a locomotive depend upon $a$ or $\mu_s$?

**Friction Wheels.** — Instead of supporting a journal on a bearing the journal may be supported upon two wheels the journals of which are in turn supported on bearings. Fig. 122 illustrates this method of supporting a journal. Here $A$ is the journal supported by the friction wheels $D$ and $E$. The bearings $B$ and $C$ of the shafts supporting these wheels rest in fixed bearings (not shown).

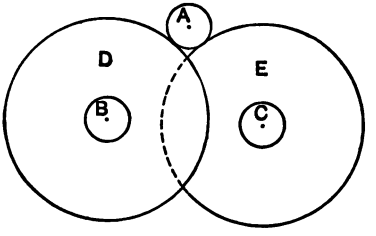


FIG. 122

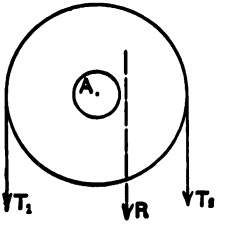


FIG. 123

As regards the bearing $A$, the journal friction, which would exist if it were supported by a bearing, is here replaced by the resistance to rolling between the journal $A$ and the wheels $D$ and $E$, in contact with which this journal rolls. The usual journal frictions of course still exist at $B$ and $C$.

If the shaft $A$ is supported in bearings (Fig. 123) and in turn supports a pulley, over which a belt passes as shown, and the tensions on both sides of this belt are equal, then the resultant force $R$, acting upon the pulley, would pass thru the center of $A$. If the bearing is frictionless the slightest increase in tension on either side of the belt would produce motion. Any increase in either tension $T_1$ or $T_2$ would shift the line of action of their resultant, $R$, to one side or other of the center of $A$.

EXERCISE 258. If the radius of the journal is $r$ and the coefficient of journal friction is $\mu_j$, how far may the line of action of $R$ in Fig. 123 be shifted without producing motion?

When the journal $A$ is supported as shown in Fig. 122 the wheels $D$ and $E$ (regarded as weightless) may be conceived as being links of the mechanism. Each will be acted upon by only two forces.

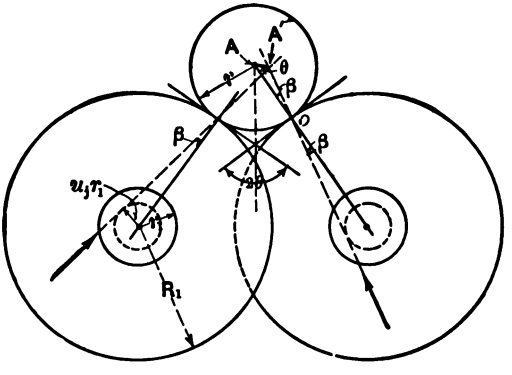


FIG. 124

EXERCISE 259. (a) Assume the mechanism, Fig. 122, frictionless and find the lines of action of the forces transmitted by the wheels $D$ and $E$ to the journal $A$.

(b) Assume journal friction at $B$ and $C$ and find the lines of action of the transmitted forces.

(c) Same as (a) or (b) but assume both journal friction and resistance to rolling.

EXERCISE 260. Show the line of action of the vertical force acting upon the shaft $A$ when motion is about to occur under the conditions (a), (b), and (c) of Ex. 259.

EXERCISE 261. Draw Fig. 122 to a large scale. Assume the same radius and coefficient of journal friction for the

journals $A$, $B$, and $C$. Determine whether the displacement of a vertical load upon the shaft $A$ must be greater or less for impending motion when $A$ is supported on a bearing or on friction wheels; thus, whether or not friction wheels reduce the friction losses at $A$.

To arrive at an approximate analytical expression for the horizontal displacement of the vertical force necessary just to overcome the frictional resistances due to a pair of friction wheels consider Fig. 124.

Neglect the resistances due to rolling, as these are very small even as compared to the radii of the friction circles involved. Then $AA'$, the required displacement, is not only very small but almost horizontal in any practical case. From the triangle $AA'O$ we have

$$\frac{AA'}{r} = \frac{\sin \beta}{\sin \theta},$$

for the angle $A'AO$ is practically equal to $\theta$, and $\beta$ is small as compared with $\theta$.

If the radius of the friction circles at the bearings of the friction wheels is $r_1\mu_j$ and the radius of the friction wheels is $R_1$,

$$\sin \beta = \frac{r_1\mu_j}{R_1},$$

so that $$AA' = r\mu_j \frac{r_1}{R_1} \frac{1}{\sin \theta}, \text{ approximately.}$$

In this expression, $r\mu_j$ is the radius of the friction circle of the main journal $A$, when resting in a bearing. Therefore, provided

$$\frac{r_1}{R_1} \frac{1}{\sin \theta} < 1,$$

the friction wheels will decrease the frictional resistances which must occur when the journal $A$ is supported by them instead of by a bearing.

It is also evident that the more nearly $2\theta$ approaches 180° (also, the larger $R_1$ and the smaller $r_1$ are made) the smaller will the frictional resistances become.

## PROBLEMS FOR REVIEW

262. The axle of a trolley car is loaded with 1000 pounds 2.5 feet to the left of the center and with 1500 pounds 2.5 feet to the right of the center. Find graphically the pressures on the rails if they lie in planes perpendicular to the axle and 2 feet from its center.

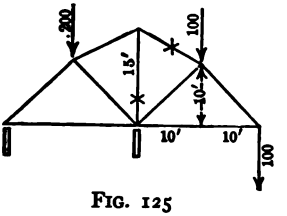


FIG. 125

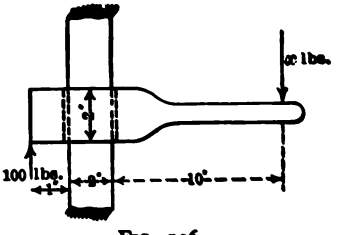


FIG. 126

263. A rope is wrapped three times around a horizontal cylinder, and a weight of 12 pounds is hung from its end.

(a) Find the least pull applied at the other end necessary to raise this weight. ($\mu = 0.3$.)

(b) What force would be required to just prevent the 12 pounds from slipping down?

264. (a) Calculate the reactions for the frame shown in Fig. 125.

(b) Calculate by the method of sections the stresses in the marked ($\times$) members.

(Measure all required distances on a large figure drawn to scale.)

265. Draw the stress diagram for the frame shown in Fig. 125.

266. The jib $AC$ of a 10-ton crane is inclined at 45° to the post, and the tie-rod $BC$ is inclined at 120° to the post.

(a) Find the stresses in the jib and tie-rod when fully loaded, the tension of the chain being neglected. (b) If a backstay $BD$ inclined at 45° to the post be added and attached to the end of a horizontal strut $AD$, find the weight at $D$ required to counterbalance the load on the crane, and find also the tension in the backstay.

267. If in Fig. 126 the arm is about to move up under the action of the 100-pound force, find graphically the greatest force $x$ which can be applied as shown. ($\mu = 0.3$.)

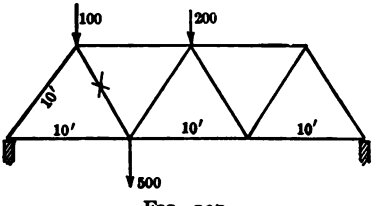


FIG. 127

268. A single fixed pulley, 6 inches in diameter, turns on an axle 2 inches in diameter ($\mu = 0.2$). A weight of 500 pounds is to be lifted by means of this pulley. Find the vertical force required.

269. A pair of sheer legs make an angle of 60° with one another and their plane is inclined at 60° to the horizontal. The backstay is inclined at 30° to the plane of the legs. Find the forces on each leg and on the backstay per ton of load carried.

270. Draw the stress diagram for the frame shown in Fig. 127. Indicate tension (+) and compression (−) on the members of the frame.

271. Find by the method of sections the stress in the marked (×) member of the frame, Fig. 127.

272. Find the reactions of the abutments graphically and draw the stress diagram for Fig. 128.

273. Find the tractive force of a locomotive having 5-foot

drivers, a 2-foot stroke, and a m.e.p. of 40 pounds per square inch with an effective piston area of 200 square inches.

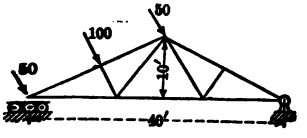


FIG. 128

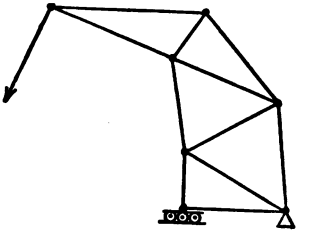

FIG. 129

274. Draw the stress diagram for the jointed frame shown in Fig. 129.

Indicate whether the various members are in tension or compression by placing the proper sign on each member.

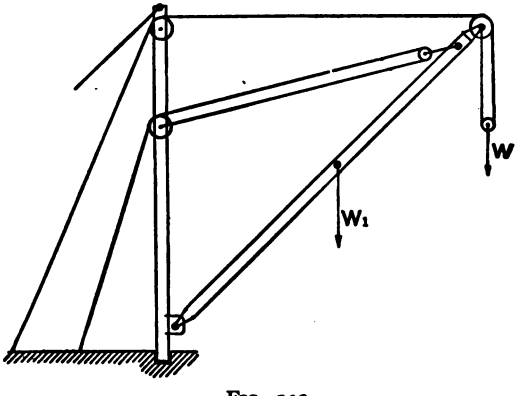


FIG. 130

275. Develop a formula for the tension in the tackle rope of the derrick shown in Fig. 130. Assume all necessary dimensions and indicate them carefully on your sketch.

276. If two turns of a rope ½ inch in diameter are taken about a post 10 inches in diameter, what force applied at

one end of the rope will just prevent a force of 659 pounds applied at the other end from producing motion? The coefficient of friction is $\frac{1}{3}$.

277. Draw the stress diagram for the frame shown in Fig. 131 and indicate whether the members are in tension or compression.

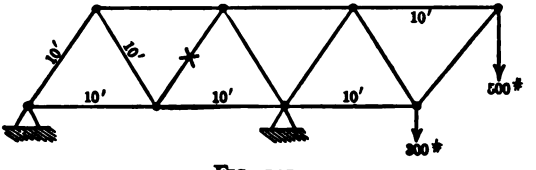


FIG. 131

278. Find the stress in the marked (×) member of the frame shown in Fig. 131 analytically by the method of sections.

279. Find graphically in the simplest manner:

(a) the reactions at $A$ and $B$,

(b) the stress in $CD$, for the crane shown in Fig. 132.

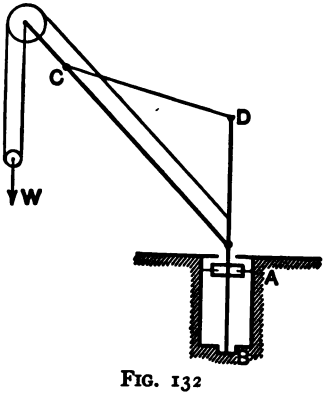


FIG. 132

280. A rope makes 2.5 turns about the wooden drum of a hoisting engine. If the load to be lifted is equivalent to 3000 pounds, at the drum what force must be applied to the rope as it leaves the drum in order to create the necessary friction? The coefficient of friction between wood and rope is 0.33.

281. During erection a force of 100,000 pounds is brought to bear at the point $A$, upon the structure illustrated in Fig. 133, by means of the cable $AB$. Compute

(a) the stress in the member marked $C$,

(b) the reactions at the abutments $M$ and $N$, due to this load.

282. Sketch enough of the stress diagram for the frame illustrated in Fig. 133 to determine the stresses in the members marked $D$ and $E$. State whether the members $D$ and $E$ are in tension or compression.

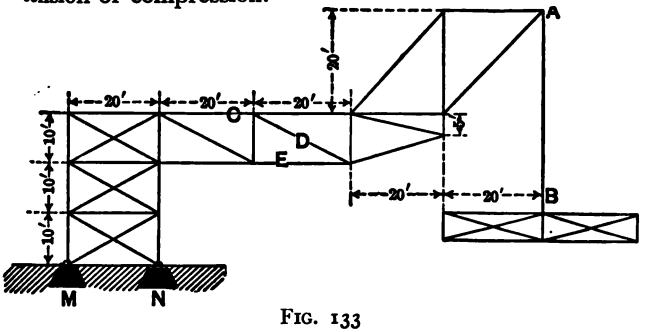


FIG. 133

283. In hemp or cotton rope-drives the greatest working stress is 140 pounds per square inch and the tension on the tight side of the rope is usually considered to be four times the tension on the slack side.

(a) What is the effective force, useful in transmitting energy, in a rope having a sectional area of 2 square inches?

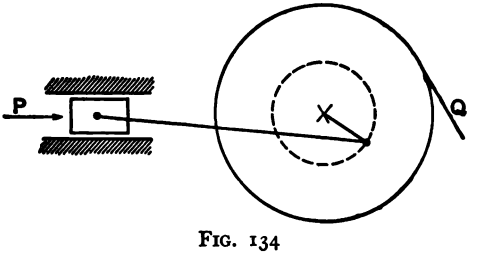


FIG. 134

(b) To what value of $\mu$ do the above tensions correspond if we assume that the common formula for belt friction applies in this case, that the angle of contact between rope and pulley is 180°, and that the rope is about to slip?

284. Find graphically the useful resistance $Q$, correspond-

ing to a given driving force $P$, in the machine shown in Fig. 134, assuming the machine frictionless.

285. Same as Ex. 284, but all frictional resistances to be taken into account.

286. A fly-wheel, weighing 2000 pounds, has a radius of gyration of 5 feet. The shaft journals are 12 inches in diameter and the coefficient of journal friction is 0.06. After the wheel reaches a speed of 120 r.p.m. the driving force is withdrawn and the wheel slows down to 60 r.p.m.

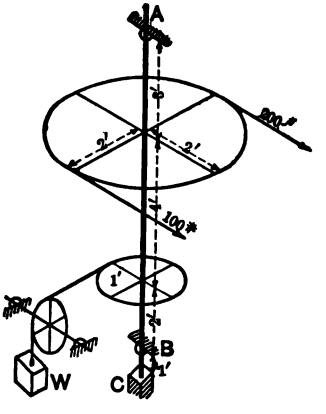


FIG. 135

(a) How much energy does the wheel lose?

(b) What moment (in pound-feet), due to friction at the shaft journals, opposes the motion of this fly-wheel?

287. Find the weight $W$, the reactions at the journals $A$ and $B$ and at the foot-step $C$ for the machine shown in Fig. 135, the machine to be considered weightless, frictionless, and in equilibrium.

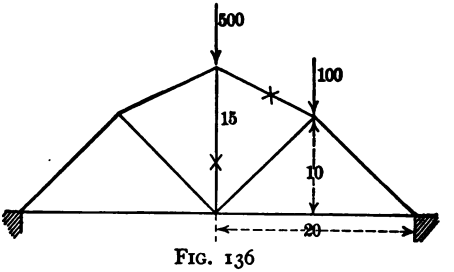


FIG. 136

288. Use the method of sections to calculate the forces in the members marked (×) in Fig. 136. Measure the necessary distances from a scale drawing.

289. If motion of the drum in Fig. 137 is impending, find the tension in each end of the strap passing around the drum. The drum is rough.

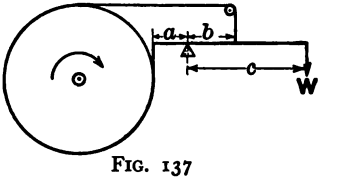

FIG. 137

290. An axle 10 feet long between bearings has 3 projecting arms of lengths 1, 3, and 1 feet; they carry masses of 2, 2, and 4 units, respectively. The arms are placed at distances of 2, 5, and 7 feet from the left bearing; the angle between the first and second is 90° and between the second and third 45°. Find the reactions of the bearings if the axle turns at 100 r.p.m.

291. A cylinder of weight $W$, moment of inertia $I$, and radius $r$ rolls down a rough plane of angle $\theta$. If the adhesion is $F$ and the coefficient of rolling resistance is $a$, find, by the method of work, the angular velocity after the cylinder rolls $s$ feet without slipping.

292. Find the friction moment of a conical pivot under the two usual assumptions.

293. The cables of a suspension bridge dip 100 feet in a span of 1000 ft. If the load is $w$ pounds per horizontal foot, find the tension in the cables at a point 200 feet horizontally from the towers.

294. A strap bearing on a brake wheel 2 feet in diameter and tightened by a lever is used to hold a load on a winch. The shaft to which the brake wheel is keyed also carries a pinion of 10 teeth gearing with a wheel of 54 teeth on a second shaft. This second shaft has a pinion of 9 teeth gearing with another wheel of 50 teeth on the drum shaft. The diameter of the drum is 12 inches, the length of the handle of the lever is 30 inches, and of the short end 3 inches. If one end of the strap, which subtends an angle of 300° at the center of the wheel, be fixed and the other end be attached to the short end of the lever, find the greatest load the

rope wound on the drum that could be supported by a force of 45 pounds applied at one end of the lever handle. Take the coefficient of friction as 0.1.

295. If a weight of 16,100 pounds attached to the rope in Ex. 294 is descending with a velocity of 300 feet per minute, how far will it go before coming to rest after the brake is put on? Neglect the kinetic energy of the rotating parts.

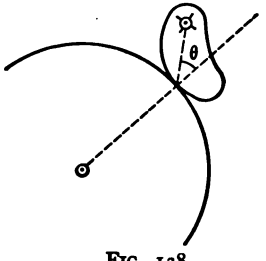


FIG. 138

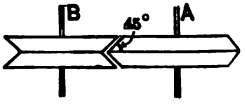


FIG. 139

296. Find the relation between $\theta$ and $\phi$, the angle of friction, so that the clutch illustrated in Fig. 138 may grip.

297. In Fig. 139 $A$ drives $B$ and 2 H.P. is transmitted with a rim velocity of 9 feet per second. If $\mu = 0.4$, with what force must the shafts at $A$ and $B$ be forced towards each other?

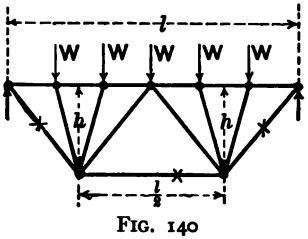


FIG. 140

298. Compute the stresses in the three members marked × in Fig. 140.

299. Draw the stress diagram for the frame shown in *Fig.* 140.

300. A locomotive running at $v$ miles per hour develops $H$ H.P.; find the tractive force, assuming no energy lost in friction.

301. The platform of a suspension foot bridge of 100 foot span is 10 feet wide and supports a load of 150 pounds per square foot. The cables have a dip of 20 feet. Find the stress in each cable at a point at a horizontal distance of 30 feet from the center of the span.

302. The fly-wheel of a shearing machine has 150,000 foot-pounds of kinetic energy stored in it when its speed is 250 r.p.m.

(a) What energy does it lose during a reduction of speed to 200 r.p.m.?

(b) If 82 % of the energy given out is imparted to the shears during a stroke of 2 inches, what is the average force, due to this energy, on the blade of the shears?

303. When the belt of the dynamometer shown in Fig. 141 is removed and the frame, pivoted at $C$, is in the

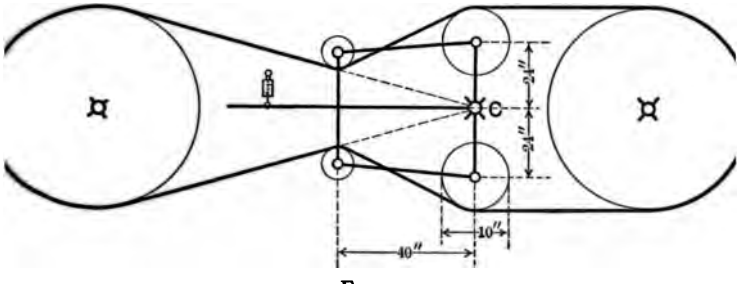


FIG. 141

position shown a spring balance at 50 inches from $C$ reads 35 pounds. When the belt travels with a velocity of 60 feet per second a weight of 100 pounds hung 40 inches from $C$ maintains the frame in its central position, the spring balance having been removed. Find the H.P. transmitted by the belt.

304. Show that the work done by water supplied to a cylinder at a constant pressure of $p$ pounds per square inch and moving a piston is $p$ times the volume of the water supplied.

305. A shaft transmits 100 H.P. and makes 200 r.p.m.; to what torque, expressed in pound-inches, is the shaft subjected?

306. A traction engine, weighing 800 pounds, has two vertical double-acting cylinders 2.5 inches in diameter and 4 inches stroke; the driving wheels, 28 inches in diameter are driven by a chain; the sprocket attached to the crank shaft is 4 inches and the one attached to the driving wheels is 12 inches in diameter. The m.e.p. is 75 pounds per square inch and 25 % of the energy developed is lost in friction. Find the available tractive force if the resistance to rolling amounts to 16 pounds per ton.

# ANSWERS.

1. $2\pi RW_2 - 2\pi rW_1$.
2. $6\pi Fr$, $10\pi Fr$ foot-pounds.
5. $\dfrac{2\pi NPl}{33{,}000}$.
6. (a) $\dfrac{33}{2\pi}$ feet $= 63.02$ inches; (b) 100.
7. $\dfrac{\pi N(2R+d)\left(\dfrac{Wb}{a}-T\right)}{12\times 33{,}000}$.
10. $\dfrac{\pi bSrN}{a(33{,}000)}$.
11. $28°\,58'$.
12. $\dfrac{(T_2-T_1)V}{33{,}000}$.
14. $\dfrac{29\pi}{66}=1.38$.
15. $\dfrac{33}{2\pi}=5\frac{1}{4}$ pounds.
17. $Wh$.
18. 199,900 foot-pounds; 4,224,000,000 foot-pounds.
19. 72,000 pounds per foot.
20. 39.4 foot-pounds.
23. $\dfrac{p_i}{r}(1+\log_e r)-p_b$.
24. 1009.
25. 1630.
26. 21.8.
29. $\dfrac{p_i(l+c)}{L+c}$; $\dfrac{p_b(c+l_1)}{c}$.
37. 163, 162 foot-pounds.
38. By t. r., 1812; By S. r., 1830 foot-pounds.
41. $PV\sin(\theta+\beta)\sin\theta$.
42. $\theta=90°-\dfrac{\beta}{2}$.

45. 15,710 pound-inches.
46. 82,500 foot-pounds.
48. $\sqrt{\frac{5}{14}}$.
49. Equal.
51. .0378 inch; 40,000 pounds; 12,940 pounds.
55. 39.28 pounds per square inch.
57. $T = \frac{d^2 pS}{2D}$ pounds.
59. $2750 + 960 = 3710$ pounds.
60. 9,790,000 foot-pounds; 297.
61. $TV = 375$ (H.P.).
62. $0.379\,Wh$ pounds; $\frac{WhV}{990}$.
64. $P = \frac{RT}{r}$.
65. $T = F$.
74. 50, $-150$; 400, 416.7; 416.7 pounds.
75. 342, 238 pounds.
76. 333, 666; 600, 2400 pounds.
78. 200 pounds.
79. $-100$ pounds.
80. $-2000$, $-667$ pounds.
81. 29 pounds.
82. 0, $-417$ pounds.
100. 7180, 5900, 1200, 5900, 6750, 22,320, 6750, 30,700 pounds.
101. 250, 1250, 500, 1250, 500, 250, 1000 pounds.
109. $-\frac{800}{9}$; $-\frac{100}{9}$, 0; $+100$, 0; $-\frac{200}{9}$, $+\frac{800}{9}$ pounds.
$-119.1$, $-29.6$; $-257.5$, $+29.6$; 388.8 pounds.
113. 349 pounds.
115. $-85.36$; $+72.7$ tons.
128. 1125, 1212 tons.
129. 450, 484 tons.
130. 5.6 inches.
131. $z = \frac{l}{\pm\sqrt{\frac{h}{h'}} + 1}$.
133. $x = m \log_e \frac{y' + s}{m}$.
134. $T = w\sqrt{m^2 + s^2} = w(y + m)$.
136. 77 feet; 60 feet.
137. 99 feet; 0.62 foot.

140. $T_{\min.} = W \sin \phi$; $\theta = \phi$.

142. $\dfrac{W \sin \phi}{\cos (\theta + \phi)}$.

144. $P_{\min.} = W \sin (\phi + \alpha)$.

145. $P = \dfrac{W \sin (\phi + \alpha)}{\cos (\theta - \phi)}$.

146. $W \tan (\phi + \alpha)$.

147. $\dfrac{W \sin (\alpha + \phi)}{\cos \phi}$.

148. $\dfrac{W \sin (\phi - \alpha)}{\cos (\phi - \theta)}$.

149. $\alpha < 2 \arctan \mu$.

150. $P_a = \dfrac{W \cos (\phi_1 + \beta) \cos (\alpha + \phi_2 + \phi_3)}{\cos \phi_3 \sin (\alpha + \phi_1 + \phi_2)}$;

$P_b = \dfrac{W \cos (\beta - \phi_1) \cos (\phi_3 + \phi_2 - \alpha)}{\cos \phi_3 \sin (\phi_1 + \phi_2 - \alpha)}$.

151. 716 pounds.

152. 140 pounds.

154. $\dfrac{P \cos \phi}{2 \sin\left(\phi + \dfrac{\alpha}{2}\right)}$.

155. $-Q\{\tan \phi + \tan (\phi - \alpha)\}$.

156. $\alpha = 2\phi$; 1 : 7.

169. 66.6%; 25%.

170. 19.5%.

171. 50%.

174. $P = \dfrac{Qr}{l} \tan (\alpha - \phi_s)$.

176. $\alpha \gtrless \phi$.

177. $\dfrac{\tan \alpha}{\tan (\alpha + \phi)}$; $\dfrac{\tan (\alpha - \phi)}{\tan \alpha}$.

181. $\alpha = 45 - \dfrac{\phi}{2}$.

183. 2.13

188. $T = 77P$; .75 inch.

189. $P' = \dfrac{T}{106} = 21.8$ pounds.

190. 3.51; 12.35; 152.4; 6611.

191. 2.2.

195. $T_c = \dfrac{12\, btv^2 W}{g}$.

197. 30 *bl.*

200. 6.65 H.P.; 96 feet per second = 5760 feet per minute.

201. 10.

203. $\frac{Rd \sin \phi}{24} \frac{2 \pi n}{33{,}000} = \frac{\pi R n d \sin \phi}{12 \times 33{,}000}$.

204. $\frac{\pi n r}{6} \frac{R \sin \phi}{\sin (\alpha + \beta)} \{\cos (\phi - \alpha) + \cos (\phi + \beta)\}$.

206. 94.5%; 96%.

207. 292, 308 pounds.

242. $\alpha = \frac{mgr}{\frac{1}{2} m_1 r^2 + 2 M r^2 + m r^2}$.

262. 1560; 940 pounds.

263. 3440; 0.042 pound.

264. − 50, + 450; 231; 206.

266. 33.5, 27.5; 23.5, 33.5 tons.

267. 12 pounds.

268. 570 pounds.

269. 2000 pounds.

271. 462.

273. 2035 pounds.

275. $\frac{W\left(d - \frac{b}{2}\right) + W_1 c}{2a}$.

276. 10 pounds.

278. 605 pounds.

280. 16.8 pounds.

281. 600,000; 400.000; 500,000 pounds.

283. 210 pounds; .44.

286. 93,100 foot-pounds; 60 pound-feet.

287. 200; 200, 100; 44.4, 155.6 pounds.

294. Two solutions depending on direction of rotation.
18,620 pounds or 11,000 pounds.
4.03 inches or 6.83 inches.

297. 215 pounds.

298. $\frac{5 W \sqrt{l^2 + 16 h^2}}{8 h}$; $\frac{3 Wl}{4 h}$.

301. 52,020 pounds.

302. 54,000 foot-pounds; 237,200 pounds.

303. 21.6.

305. 31,500 pound-inches.

306. 144 pounds.

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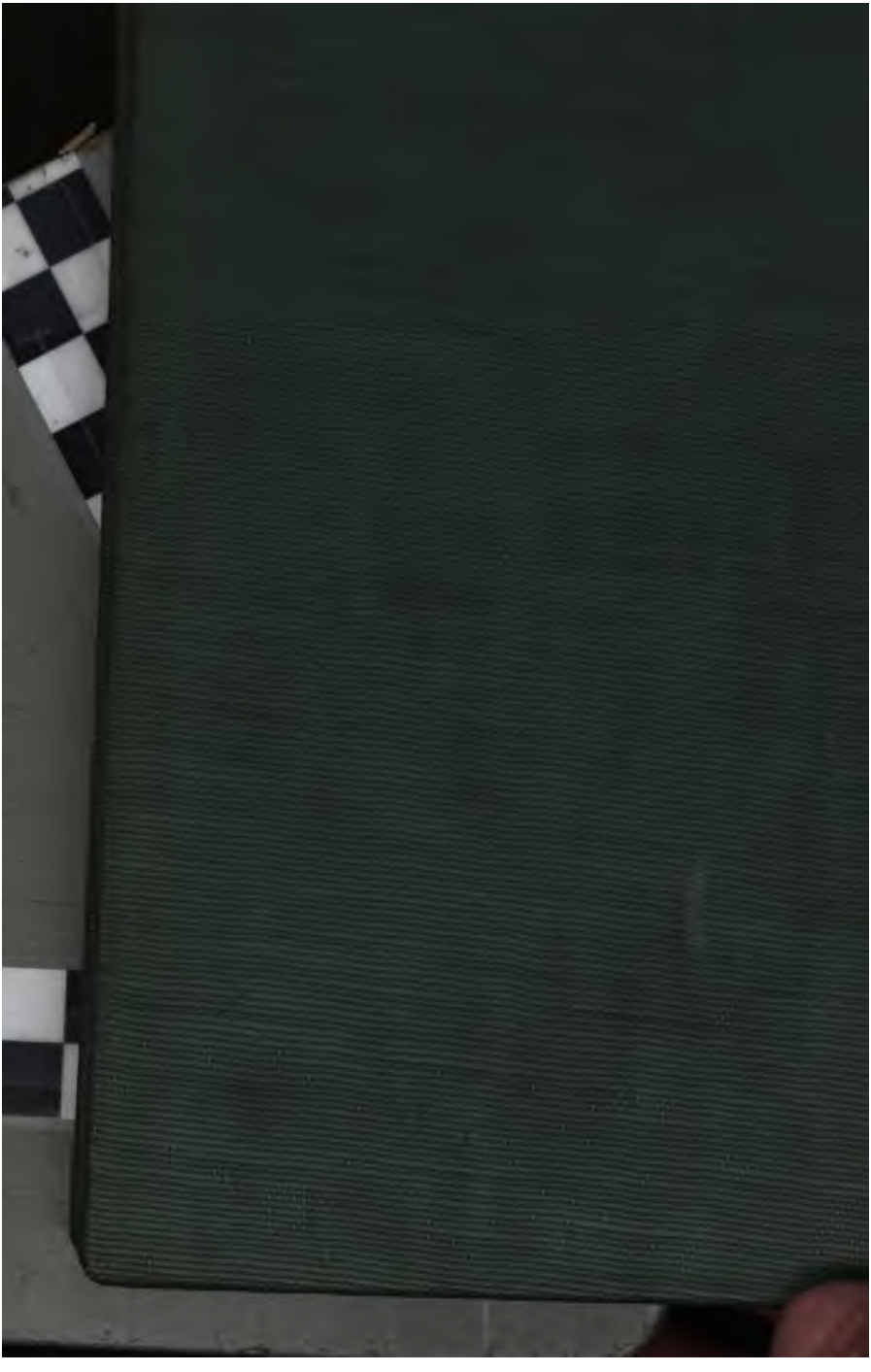
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